

**DEFINABLE TOPOLOGICAL SPACES IN O-MINIMAL  
STRUCTURES**

by

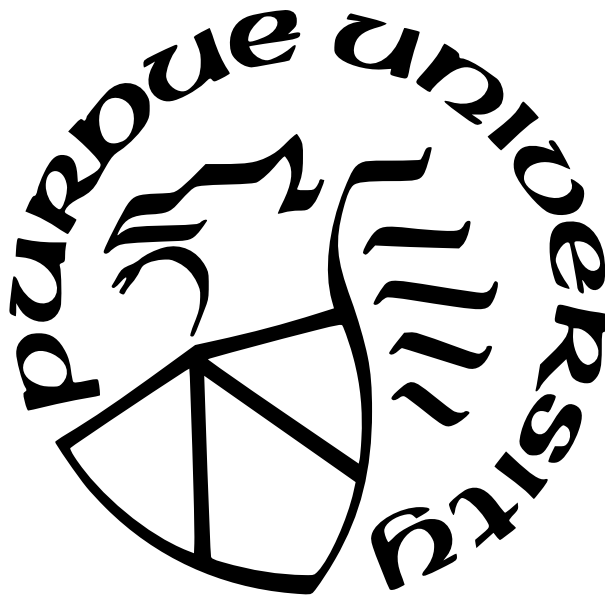
**Pablo Andújar Guerrero**

**A Dissertation**

*Submitted to the Faculty of Purdue University*

*In Partial Fulfillment of the Requirements for the degree of*

**Doctor of Philosophy**



Department of Mathematics

West Lafayette, Indiana

August 2021

**THE PURDUE UNIVERSITY GRADUATE SCHOOL  
STATEMENT OF COMMITTEE APPROVAL**

**Prof. Margaret E. M. Thomas (Chair)**

Department of Mathematics, Purdue University

**Prof. Matthias Aschenbrenner**

Department of Mathematics, University of Vienna

**Prof. Saugata Basu**

Department of Mathematics, Purdue University

**Prof. Andrei Gabriellov**

Department of Mathematics, Purdue University

**Approved by:**

Prof. Plamen D Stefanov

A mis padres y a mi tía María Dolores.

## ACKNOWLEDGMENTS

I would like to thank my parents, who will not understand a single word in this thesis, but are prouder than anyone that it exists.

A special thank you also goes to Erik Walsberg. This thesis started, in a way, as a continuation of his own PhD thesis. He has done a great deal in shaping its contents, in particular by co-authoring papers on which Chapters 4 and 6 are based. I thank him for his many ideas, his guidance and his support. He is a cool guy.

My gratitude is also owed to the following people. To Sergei Starchenko, for valuable suggestions that helped develop the contents of Chapter 5. To Patrick Speissegger and the logic group at McMaster University, for their generosity. To Will Johnson, for helpful conversations on o-minimal types, and for providing an alternative proof to Theorem 5.2.11 and a counterexample to a parameter version of Lemma 5.2.6. To the Nguyen-Hellmann family, Diana, Rosemarie and An, for treating me as one of their own. To Sebastian Krapp, for doing more than he knows.

During my years as doctoral researcher I received funding support from the German Research Council (DFG), Grant TH 1781/2-1; the Zukunftskolleg, Universität Konstanz; the Canada Natural Sciences and Engineering Research Council (NSERC), Discovery Grant RGPIN-06555-2018; and the Department of Mathematics at Purdue University.

Finally, I thank my supervisor, Margaret Thomas, for uncountably many hours of reading and correcting my work, for all the advice and the never ending encouragement, and for teaching me more than how to write a PhD thesis in mathematics. Above all, I thank her for sticking with me through strange and difficult times. Without her I would have not done this.

# TABLE OF CONTENTS

LIST OF FIGURES . . . . .	8
ABSTRACT . . . . .	9
1 INTRODUCTION . . . . .	11
2 DEFINITIONS AND PRELIMINARY RESULTS . . . . .	21
2.1 General conventions and preliminary results . . . . .	21
2.1.1 The euclidean topology . . . . .	22
2.1.2 O-minimality . . . . .	22
2.1.3 Intersecting families of sets . . . . .	23
2.1.4 Types and tame extensions . . . . .	24
2.2 Definable topological spaces . . . . .	26
2.2.1 Definable metric spaces . . . . .	28
2.2.2 Definable connectedness and the frontier dimension inequality . . . . .	29
2.2.3 Definable curves . . . . .	30
2.2.4 Push-forwards . . . . .	32
2.2.5 $\epsilon$ -Accummulation sets . . . . .	33
3 DEFINABLE SEPARABILITY . . . . .	36
3.1 Definitions and basic results . . . . .	36
3.2 Main result . . . . .	39
4 DEFINABLE FIRST COUNTABILITY . . . . .	48
4.1 Directed sets and families . . . . .	49
4.2 Main result on directed sets . . . . .	51
4.3 A strengthening of the main result for expansions of ordered fields . . . . .	60
4.4 Tame Extensions . . . . .	62
4.5 Definable first countability . . . . .	67
5 DEFINABLE COMPACTNESS . . . . .	73

5.1	Preliminaries . . . . .	75
5.1.1	Preorders induced by types . . . . .	75
5.1.2	Transversals and intersection properties . . . . .	76
5.1.3	VC theory . . . . .	77
5.2	Types . . . . .	78
5.3	Transversals . . . . .	89
5.3.1	Forking, dividing and definable types . . . . .	103
5.4	Definable compactness . . . . .	106
5.4.1	Definitions and basic facts . . . . .	106
5.4.2	Characterizing definable compactness . . . . .	109
5.A	Proof of The Marker-Steinhorn Theorem . . . . .	119
6	ONE-DIMENSIONAL DEFINABLE TOPOLOGICAL SPACES . . . . .	124
6.1	Definitions and first examples . . . . .	125
6.2	Basic results . . . . .	127
6.3	$T_1$ and Hausdorff ( $T_2$ ) spaces. Decomposition in terms of the $\tau_e$ , $\tau_c$ , $\tau_r$ and $\tau_l$ topologies . . . . .	137
6.4	Hausdorff regular ( $T_3$ ) spaces. Decomposition in terms of the $\tau_{lex}$ and $\tau_{Alex}$ topologies. . . . .	144
6.5	Some universality results . . . . .	156
6.6	Definable Hausdorff compactifications . . . . .	164
6.7	Affine one-dimensional topologies . . . . .	170
6.8	Definable metrizability . . . . .	174
6.9	An affineness result by Peterzil and Rosel . . . . .	179
7	GENERAL AFFINENESS RESULTS . . . . .	181
7.1	Affineness results by van den Dries and Walsberg . . . . .	181
7.1.1	Van den Dries' definable manifold spaces . . . . .	181
7.1.2	Walsberg's definable metric spaces . . . . .	185
	A proof of Theorem 7.1.5 using the definable Tietze Extension Theorem	188
	An elementary proof of Theorem 7.1.5 . . . . .	190

7.2 An affineness result for definably Tychonoff spaces . . . . .	193
A EXAMPLES . . . . .	211
VITA . . . . .	221
REFERENCES . . . . .	222

## LIST OF FIGURES

5.1	Depicting (in blue) the set $A(x', x'', x''', y', y'', y''')$ . . . . .	118
-----	---	-----



## ABSTRACT

We further the research in o-minimal topology by studying in full generality definable topological spaces in o-minimal structures. These are topological spaces  $(X, \tau)$ , where  $X$  is a definable set in an o-minimal structure and the topology  $\tau$  has a basis that is (uniformly) definable. Examples include the canonical o-minimal “euclidean” topology, “definable spaces” in the sense of van den Dries [17], definable metric spaces [49], as well as generalizations of classical non-metrizable topological spaces such as the Split Interval and the Alexandrov Double Circle.

We develop a usable topological framework in our setting by introducing definable analogues of classical topological properties such as separability, compactness and metrizability. We characterize these notions, showing in particular that, whenever the underlying o-minimal structure expands  $(\mathbb{R}, <)$ , definable separability and compactness are equivalent to their classical counterparts, and a similar weaker result for definable metrizability. We prove the equivalence of definable compactness and various other properties in terms of definable curves and types. We show that definable topological spaces in o-minimal expansions of ordered groups and fields have properties akin to first countability. Along the way we study o-minimal definable directed sets and types. We prove a density result for o-minimal types, and provide an elementary proof within o-minimality of a statement related to the known connection between dividing and definable types in o-minimal theories.

We prove classification and universality results for one-dimensional definable topological spaces, showing that these can be largely described in terms of a few canonical examples. We derive in particular that the three element basis conjecture of Gruenhage [25] holds for all infinite Hausdorff definable topological spaces in o-minimal structures expanding  $(\mathbb{R}, <)$ , i.e. any such space has a definable copy of an interval with the euclidean, discrete or lower limit topology.

A definable topological space is affine if it is definably homeomorphic to a euclidean space. We prove affineness results in o-minimal expansions of ordered fields. This includes a result for Hausdorff one-dimensional definable topological spaces. We give two new proofs of an affineness theorem of Walsberg [49] for definable metric spaces. We also prove an affineness

result for definable topological spaces of any dimension that are Tychonoff in a definable sense, and derive that a large class of locally affine definable topological spaces are affine.

# 1. INTRODUCTION

O-minimality is envisioned as a far reaching generalization of real semialgebraic and subanalytic geometry, isolating the tameness of these theories while having ample applicability. After the seminal insights of van den Dries [16] [14], the definition of o-minimal structure was coined by Pillay and Steinhorn [40]. That is, a model theoretic structure is o-minimal if it expands a linear order  $(M, <)$  and every (parametrically) definable subset of  $M$  is a finite union of points and intervals with endpoints in  $M \cup \{-\infty, \infty\}$ . Since then, the vision of o-minimality has been realized in two ways. On the one hand, some prominent structures expanding the ordered field of reals have been shown to be o-minimal. These include, for example, expansions by the real exponential function [51] and by the real exponential function and restricted real analytic functions [19]. On the other hand, the study of mathematical objects definable in o-minimal structures has become a fruitful area of research. Parting from sets and functions with the canonical o-minimal “euclidean” topology, this research has expanded into groups, fields, topological families of functions, metrics and linear orders among others, with their study reaching into areas such as topology, geometry, algebra, combinatorics, machine learning and economics. An underlying thesis motivating this program is the idea that o-minimality provides a mathematical framework for the theory of these objects that is both rich and tame. In this sense, van den Dries [17] proposed o-minimality as a framework for the “topologie modérée” (tame topology) of Grothendieck [24], a setting for topology and geometry that avoids pathological objects. In this thesis we focus on the capabilities of o-minimality as a tame topological setting.

The work of Knight, Pillay and Steinhorn [29] established, through the cell decomposition theorem, the topological tameness of definable sets and functions with the o-minimal euclidean topology. Pillay [39] proved that definable groups in o-minimal structures have a natural definable manifold topology that makes them topological groups. This motivated the study of topologies within o-minimality beyond the canonical euclidean one. Van den Dries [17] applied techniques from semialgebraic topology to study these manifold spaces. He showed that, in an o-minimal expansion of a field, if a manifold space is regular then it is definably homeomorphic to a euclidean space (we call this being affine). Van den Dries

also studied limit sets in o-minimal expansions of the field of reals [15], proving definability results for the Hausdorff and Gromov-Hausdorff metrics and the Tychonov topology among definable families of sets. In the next years Thomas [48], motivated by the proof of the Reparametrization Theorem of Pila and Wilkie, an application of o-minimality to diophantine geometry, studied convergence results among definable families of functions with  $C^r$ -norms and norms involving a Lipschitz constant. She proposed the development of a theory of definable normed spaces within o-minimality. This was generalized by Walsberg [49], who introduced and studied definable metric spaces in o-minimal expansions of fields. He proved a strong affineness result, showing in particular that any separable definable metric space in an o-minimal expansion of the field of reals is affine. Definable orders in o-minimal structures, which encapsulate definable order topologies, were studied by Ramakrishnan and Steinhorn ([42], [43]), with applications to economics. Recently, Johnson [27] studied interpretable sets, proving that their natural quotient topology is piecewise affine in a strong sense.

In this thesis we push further the vision of o-minimality as a tame topological setting, and expand on the work of Walsberg [49], by studying in full generality topologies that are (explicitly) definable in o-minimal structures in the sense of Flum and Ziegler [22] and Pillay [38]. In other words, we study topologies that admit a (uniformly) definable basis.

**Definition.** *Let  $\mathcal{M} = (M, <, \dots)$  be an o-minimal structure. A definable topological space in  $\mathcal{M}$  is a topological space  $(X, \tau)$  such that  $X \subseteq M^n$ , for some  $n < \omega$ , and there exists a definable (possibly with parameters) family of sets that is a basis for  $\tau$ . We say that the topology  $\tau$  is definable.*

Our aim is to develop an o-minimal theory of general topology. We build this topological landscape by either introducing or adapting from the literature properties expressible in first order logic that serve as analogues of classical topological notions such as separability, compactness and metrizability. We study these properties, both to justify their suitability as analogues of their classical counterparts, and as important tools in the classification and study of definable topological spaces. In the particular case of our characterization of definable compactness, we seek to unify in the o-minimal setting previous distinct definitions of the notion. We carry out our topological research within this framework.

We now describe the main results in this thesis.

We devote Chapters 3, 4 and 5 to the construction of our topological landscape through the study of various definable topological properties. In Chapter 3 we define the notion of definable separability to be the property that a space does not admit an infinite definable family of pairwise disjoint open sets (note the similarity with the countable chain condition). We prove (Proposition 3.1.6) that this is equivalent, among definable metric spaces, to the namesake notion due to Walsberg [49] (the non-existence of an infinite definable discrete subspace), which in turn is not suitable for general definable topological spaces (see Example A.9). We further characterize definable separability, proving that, for topological spaces definable in o-minimal expansions of  $(\mathbb{R}, <)$ , the notion is equivalent to both the countable chain condition and classical separability (Theorem 3.2.6).

The following is the main result in Chapter 4. We refer to it loosely as “definable first countability”. It allows us to conclude that, in o-minimal expansions of ordered fields, definable topological spaces display properties akin to first countability, in particular in the sense that definable curves (as analogues of sequences) play a crucial role in describing the topology. The full result includes a suitable analogue in the case of an o-minimal expansion of an ordered group (Corollary 4.5.3 (1)). For clarity below we only state the field case.

**Theorem A** (Corollary 4.5.3 (2) and Proposition 4.5.4). *Let  $\mathcal{M}$  be an o-minimal expansion of an ordered field. Let  $(X, \tau)$  be a definable topological space in  $\mathcal{M}$ . Then, for every  $x \in X$ , there exists a definable basis of open neighborhoods of  $x$  in  $(X, \tau)$  of the form*

$$\{A_t : t > 0\}$$

*satisfying that, for every  $0 < s < t$ ,  $A_s \subseteq A_t$ .*

*It follows that  $(X, \tau)$  has definable curve selection.*

The proof of Theorem A is based on the study of definable directed sets in o-minimal expansions of ordered groups and fields, in particular the proof that these admit certain definable cofinal maps (Theorem 4.2.2 and Corollary 4.3.4).

One of the main aims of this thesis, which we realize in Chapter 5, is the study of the following notion of definable compactness: every downward directed definable family of (nonempty) closed sets has nonempty intersection. We choose a definition of definable compactness in terms of closed sets to establish a parallelism with classical compactness. Notably, our definition is different from the original namesake notion in terms of converging definable curves introduced by Peterzil and Steinhorn in the context of definable manifold spaces [37], which we denote definable curve-compactness. Ultimately, we justify our definition by proving that definable and classical compactness are equivalent whenever the underlying o-minimal structure expands the reals (Theorem E (2)), an equivalence that fails for definable curve-compactness.

We approach the study of definable compactness by analyzing definable downward directed families of sets in general (Definition 2.1.6), in connection with definable types. Inspired by our work on definable directed sets used to prove Theorem A, we reach the following strong density result for types with a basis given by a definable (downward directed) family of cells.

**Theorem B** (Theorem 5.2.11 (definable case)). *Let  $\mathcal{S} \subseteq \mathcal{P}(M^n)$  be a definable family of sets in an o-minimal structure  $\mathcal{M}$ . The following are equivalent.*

- (1)  $\mathcal{S}$  extends to a definable  $n$ -type.
- (2) There exists a definable downward directed family of (nonempty) sets  $\mathcal{F}$  such that, for every  $S \in \mathcal{S}$ , there is some  $F \in \mathcal{F}$  with  $F \subseteq S$ .
- (3)  $\mathcal{S}$  extends to an  $n$ -type with a basis given by a definable family of cells.

Our approach in Theorem B to study o-minimal types seems to be novel.

Still with the aim of characterizing definable compactness, we proceed by investigating intersection properties among definable families of sets in connection with definable types. We give an elementary proof within o-minimality of the following fact.

**Theorem C** (Theorem 5.3.9 and Proposition 5.3.18). *Let  $\mathcal{S}$  be a definable family of nonempty sets in an o-minimal structure  $\mathcal{M}$ . Let  $q = \max\{1, \dim \cup \mathcal{S}\}$ . The following are equivalent.*

- (1) *There exists some  $p \geq q + 1$  such that  $\mathcal{S}$  has the  $(p, q + 1)$ -property, i.e. for every  $p$  sets in  $\mathcal{S}$ , some  $q + 1$  intersect.*
- (2)  *$\mathcal{S}$  can be covered by finitely many subfamilies, each of which extends to a complete definable type. If  $\mathcal{S}$  is definable over  $A \subseteq R$  then the types can be chosen  $A$ -definable.*

We also prove a version of Theorem C in terms of VC-codensity (Corollary 5.3.11). We observe, with the use of VC theory, how these results are equivalent to the following fact, known in o-minimal theories and a more general class of dp-minimal theories [47]: a formula does not divide (equivalently does not fork) over a model  $M$  if and only if it extends to an  $M$ -definable type. Our proof of Theorem C is elementary in that it avoids the use of any forking or VC literature. We derive (Theorem 5.3.16) that the aforementioned equivalence between formulas not dividing and extending to definable types holds within o-minimality for dividing over any set (not just a model).

Finally we are able to characterize definable compactness through the next Theorem D. In particular we use Theorem B to show its equivalence with specialization-compactness (2), a notion introduced by Fornasiero [23], and Theorem C to prove the equivalence with (5). We also describe the relationship between compactness and definable curve-compactness (7).

**Theorem D** (Theorem 5.4.9). *Let  $\mathcal{M}$  be an o-minimal structure. Let  $(X, \tau)$  be a definable topological space in  $\mathcal{M}$ . The following are equivalent.*

- (1)  *$(X, \tau)$  is definably compact, i.e. every downward directed definable family of (nonempty) closed sets has nonempty intersection.*
- (2) *Every definable complete type in  $X$  has a specialization, i.e. there is a point contained in every closed set in the type.*
- (3) *Any definable family of  $\tau$ -closed sets that extends to a (complete and global) definable type has nonempty intersection.*
- (4) *Any definable family of closed sets with the finite intersection property has a finite transversal, i.e. there exists a finite set that intersects every set in the family.*

(5) Any definable family  $\mathcal{C}$  of closed sets with the  $(p, q)$ -property, where  $p \geq q > \max\{1, \dim \cup \mathcal{C}\}$ , has a finite transversal.

(6) Any definable family  $\mathcal{C}$  of closed sets with the  $(p, q)$ -property, where  $p \geq q$  and  $q$  is greater than the VC-codensity of  $\mathcal{C}$ , has a finite transversal.

Moreover all the above imply and, if  $\tau$  is Hausdorff or  $\mathcal{M}$  has definable choice, are equivalent to:

(7)  $(X, \tau)$  is definably curve-compact, i.e. every definable curve in  $X$  converges.

Theorem D generalizes the work of Peterzil and Pillay [35], who proved the equivalence (1) $\Leftrightarrow$ (4) for definable families of definably compact sets in the euclidean topology in o-minimal structures with definable choice. Interestingly, they extracted their proof from the work of Dolich [12] on the connection between forking and definable types in o-minimal theories, highlighting once again the link between o-minimal stability theory and topology.

Our work leading to Theorem D can be used to expand on the definable Helly's Theorem of Aschenbrenner and Fischer [5]. We observe in particular that, in an o-minimal expansion of an ordered field  $\mathcal{M}$ , any family of convex subsets of  $M^n$  with the property that every subfamily of size  $n + 1$  has nonempty intersection extends to a definable type (Remark 5.4.10 (ii)).

The theorem below collects our results on the equivalence between definable topological properties and their classical counterparts in o-minimal expansions of the reals. The work on definable metrizability, a notion imported from the work of Walsberg [49], is described in Chapter 6, Section 6.8.

**Theorem E.** *Let  $(X, \tau)$  be a definable topological space in an o-minimal expansion  $\mathcal{M}$  of  $(\mathbb{R}, <)$ . The following hold.*

(1)  $(X, \tau)$  is definably separable if and only if separable.

(2)  $(X, \tau)$  is definably compact if and only if compact.

If  $\mathcal{M}$  expands the field of reals and  $\dim X \leq 1$ , then the following also holds.



(3)  $(X, \tau)$  is definably metrizable if and only if metrizable.

Theorem E justifies our definitions of definable topological properties, and moreover shows that classical topological properties can be captured by first order logic, in the setting of an o-minimal expansion of the reals.

The equivalence described in Theorem E (2), which corresponds to Corollary 5.4.14, is proved using specialization-compactness (Theorem D(2)) and the Marker-Steinhorn Theorem (Theorem 2.1.8). On the other hand, (1) and (3) in Theorem E, which correspond respectively to Theorems 3.2.6 and 6.8.2, follow from elementary proofs. It is open whether or not (3) holds in greater generality.

In Chapter 6 we proceed with a detailed study of one-dimensional definable topological spaces. Among the examples of such spaces we observe some rather classical topological spaces that had not previously been considered in an o-minimal context. It is perhaps not surprising that spaces such as the Sorgenfrey Line, Split Interval and the Alexandrov Double Circle (Examples A.3, A.4 and A.13 respectively), which were defined during the onset of topology as counterexamples to generalizations of metric and euclidean topology to general topology, are definable in the field of reals.

On the other hand, we find strong dividing lines among one-dimensional o-minimal definable topologies. We note, for example, that the Cantor space is not definable (Corollary 6.3.8), and prove strong decomposition and universality theorems.

Our main decomposition result is Theorem F below. It is related to the 3-element basis conjecture of Gruenhage [25], which is an open conjecture in set-theoretic topology stating that ZFC and the proper forcing axiom imply the following property ( $\star$ ): every uncountable first countable regular Hausdorff topological space contains a subspace of cardinality  $\aleph_1$  with either the euclidean, lower limit or discrete topologies. We prove (Theorem F (1)) a decomposition result stronger than ( $\star$ ) for definable topological spaces  $(X, \tau)$  in  $\mathcal{M}$  where  $X \subseteq M$  (i.e. spaces in the line). Along the way we note that ( $\star$ ) holds for all infinite  $T_1$  definable topological spaces (Remark 6.3.3). We improve our decomposition result in the case of Hausdorff regular spaces in the line (Theorem F (2)).

**Theorem F** (Theorems 6.3.9 and 6.4.3). *Let  $\mathcal{M} = (M, <, \dots)$  be an o-minimal structure. Let  $(X, \tau)$  be a definable topological space in  $\mathcal{M}$  with  $X \subseteq M$ . Let  $0 < 1 < 2 < \dots$  be fixed constants.*

- (1) *If  $(X, \tau)$  is Hausdorff then it can be partitioned into finitely many points and intervals such that, on each interval, the  $\tau$  subspace topology is either euclidean, discrete, or the lower or upper limit topologies.*
- (2) *If  $(X, \tau)$  is Hausdorff and regular then it can be partitioned into three definable sets  $A$ ,  $B$  and  $F$  such that  $F$  is finite and  $A$  and  $B$  are open, and moreover*
  - (i) *there exists some  $n < \omega$  such that  $A$  embeds definably into the space  $M \times \{0, \dots, n\}$  with the lexicographic order topology;*
  - (ii) *there exists some  $n < \omega$  such that  $B$  embeds definably into the definable Alexandrov  $n$ -line (Example A.5), a generalization of the classical Alexandrov Double Circle space.*

From Theorem F we conclude that, even though there exist one-dimensional definable topological spaces displaying a wide variety of distinct topological properties, a few classical examples describe large classes of these spaces, whose overall structure is ultimately quite restrictive. This fails to be true for spaces of larger dimension, which display wilder behaviour, as discussed in Appendix A, where we list a number of examples.

Finally, we consider the question of affineness for definable topological spaces. Recall that a space is affine if it is definably homeomorphic to a set with the euclidean topology. Our aim is to build on the literature on affineness results, in particular the theorems of van den Dries for definable manifold spaces (Theorem 7.1.2) and of Walsberg for definable metric spaces (Theorem 7.1.5). We start by proving the following affineness theorem for one-dimensional definable topological spaces. The research on affineness is done in the context of o-minimal expansions of ordered fields.

**Theorem G** (Theorem 6.7.1). *Let  $\mathcal{M}$  be an o-minimal expansion of an ordered field. Let  $(X, \tau)$ ,  $\dim X \leq 1$ , be a Hausdorff definable topological space in  $\mathcal{M}$ . Exactly one of the following holds:*

- (1)  $(X, \tau)$  contains a subspace definably homeomorphic to an interval with either the discrete or the lower limit topology;
- (2)  $(X, \tau)$  is affine.

We compare Theorem G to the main result of Peterzil and Rosel [36] (Theorem 6.9.1), who also recently studied one-dimensional definable topological spaces in o-minimal structures. We establish the equivalence between their affineness result and ours. We also answer some of their open questions (see Section 6.9).

We devote Chapter 7 to the investigation of affineness results for spaces of all dimensions. We survey the proof of the corresponding theorems by van den Dries and Walsberg, giving two new proofs of the theorem of the latter, one using the definable Tietze extension theorem for definable metric spaces (Theorem 7.1.10), and another more elementary proof (see Section 7.1.2). We conclude by proving the following affineness theorem for spaces of all dimensions.

**Theorem H** (Theorem 7.2.4 and Corollary 7.2.18). *Let  $\mathcal{M}$  be an o-minimal expansion of an ordered field. Let  $(X, \tau)$  be a definable topological space in  $\mathcal{M}$  satisfying the following three conditions.*

- (1)  $(X, \tau)$  is definably compact.
- (2)  $(X, \tau)$  has the frontier dimension inequality, i.e. any definable subset  $Y \subseteq X$  satisfies  $\dim(\partial_\tau Y) < \dim Y$ , where  $\partial_\tau Y$  denotes the  $\tau$ -frontier of  $Y$ .
- (3)  $(X, \tau)$  is Hausdorff and there exists a definable family of continuous functions  $X \rightarrow M$  such that their induced weak topology on  $X$  equals  $\tau$ .

Then  $(X, \tau)$  is affine.

In particular any locally affine Hausdorff space satisfying (1) and (2) is affine, where locally affine is the property that every point has an affine neighborhood.

In order to prove Theorem H we use an approach from functional analysis, in particular the proof that a compact Hausdorff space is metrizable if and only if its space of scalars is

separable, as well as Walsberg's affineness result (Theorem 7.1.5). This allows us to transform the question into an investigation of the metrizability of the pointwise convergence topology among certain definable families of continuous functions. We then proceed using tools of o-minimality.

In assessing any improvement to Theorem H we find limitations in the fact that it is not clear whether or not definable versions of Urysohn's lemma or the Tietze extension theorem hold in the o-minimal setting.

Chapter 4 is based on the author's publication with Margaret Thomas and Erik Walsberg [2]. Chapter 6 is based on a further paper in preparation with the same authors.

## 2. DEFINITIONS AND PRELIMINARY RESULTS

### 2.1 General conventions and preliminary results

Throughout this thesis we fix an o-minimal expansion  $\mathcal{M} = (M, <, \dots)$  of a dense linear order without endpoints, i.e.  $\mathcal{M}$  a linearly densely ordered structure satisfying that any definable subset of  $M$  is a finite union of points and intervals with endpoints in  $M \cup \{-\infty, +\infty\}$ . Unless stated otherwise, by “definable” we mean “definable in  $\mathcal{M}$ , possibly with parameters from  $M$ ”. We denote the algebra of definable subsets of  $M^n$ , for some  $n < \omega$ , by  $Def(M^n)$ . At times we assume that  $\mathcal{M}$  expands an ordered group  $(M, 0, +, <)$  or field  $(M, 0, 1, +, \cdot, <)$ . In chapters and sections in which we assume throughout that  $\mathcal{M}$  expands an ordered field we use notation  $\mathcal{R}$  and  $R$  in place of  $\mathcal{M}$  and  $M$  respectively.

Throughout  $n, m, k$  and  $l$  denote natural numbers. All variables and parameters  $u, x, a \dots$  are  $n$ -tuples. Let  $l(u)$  denote the length of any given variable or parameter  $u$ .

Every formula we consider is in the language of  $\mathcal{M}$ . Let  $\varphi(u, v)$  and  $\phi(v)$  be formulas in  $m+n$  and  $n$  free variables respectively. For any set  $S \subseteq M^n$  and  $a \in M^m$ , let  $\varphi(a, S) = \{b \in S, \mathcal{M} \models \varphi(a, b)\}$  and  $\phi(S) = \{b \in S : \mathcal{M} \models \phi(b)\}$ . We say that a family of sets  $\mathcal{S} \subseteq \mathcal{P}(M^n)$  is definable when it is uniformly definable, meaning that there exists some definable index set  $\Omega \subseteq M^m$  and some formula  $\varphi(u, v)$  in  $m+n$  free variables such that  $\mathcal{S} = \{\varphi(a, M^n) : a \in \Omega\}$ .

Generally we will use notation  $\mathcal{S}, \mathcal{X}, \mathcal{Y}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{F}, \mathcal{G}$  and  $\mathcal{H}$  to refer to definable families of sets. In particular, we often use  $\mathcal{D}$  for cell partitions,  $\mathcal{C}$  for families of closed sets, and  $\mathcal{F}, \mathcal{G}$  and  $\mathcal{H}$  for families of functions.

For a given  $n$ , let  $\pi$  denote the projection  $M^{n+1} \rightarrow M^n$  to the first  $n$  coordinates, where  $n$  will often be omitted and clear from context. For a family  $\mathcal{S} \subseteq \mathcal{P}(M^{n+1})$  let  $\pi(\mathcal{S}) = \{\pi(S) : S \in \mathcal{S}\}$ .

Let  $M_{\pm\infty} = M \cup \{-\infty, +\infty\}$ . Unless stated otherwise, by interval we mean an open interval with respect to the order  $<$  and with distinct endpoints in  $M_{\pm\infty}$ .

Given a function  $f$  we denote its domain  $dom(f)$  and its graph by  $graph(f)$ . We adopt the convention of saying that a function  $f : M^n \rightarrow M_{\pm\infty}$  is definable if its restriction  $f|_{f^{-1}(M)}$  and the sets  $f^{-1}(+\infty)$  and  $f^{-1}(-\infty)$  are definable, and similarly say that a family

$\mathcal{F} = \{f_u : u \in \Omega\}$  of functions  $M^n \rightarrow M_{\pm\infty}$  is definable if the families  $\{f_u|_{f_u^{-1}(M)} : u \in \Omega\}$ ,  $\{f_u^{-1}(+\infty) : u \in \Omega\}$  and  $\{f_u^{-1}(-\infty) : u \in \Omega\}$  are.

### 2.1.1 The euclidean topology

By the euclidean topology on  $M^n$  we refer to the canonical topology in an o-minimal structure, which is given by the order topology when  $n = 1$  and its induced product topology when  $n > 1$ . We interpret the euclidean topology in  $(M_{\pm\infty})^n$  to be its natural extension.

Without reference to a particular topology, any topological notion is to be understood with respect to the euclidean topology.

### 2.1.2 O-minimality

For most of the basics of o-minimality, including the full cell decomposition theorem, we direct the reader to [17]. We present some basic facts and notation, starting with uniform cell decomposition.

Regarding cells, we use the following standard notation. For any set  $C$  and any two functions  $f, g : C \rightarrow M_{\pm\infty}$ , with  $f < g$ , let  $(f, g)_C = \{\langle x, t \rangle \in C \times M_{\pm\infty} : f(x) < t < g(x)\}$ . We sometimes omit  $C$  if it is clear from context and write simply  $(f, g)$ .

Recall that a *cell decomposition*  $\mathcal{D}$  of a definable set  $S \subseteq M^n$  is a finite partition of  $S$  into cells such that, for any  $0 < m < n$ , the family  $\{\pi_{\leq m}(D) : D \in \mathcal{D}\}$  is a cell partition of  $\pi_{\leq m}(S)$ , where  $\pi_{\leq m} : M^n \rightarrow M^m$  denotes the projection to the first  $m$  coordinates. By o-minimality every definable set admits a cell decomposition.

**Proposition 2.1.1** (Uniform cell decomposition, [17], Chapter 3, Proposition 3.5). *Let  $S \subseteq M^{n+m}$  be a definable set and let  $\mathcal{D}$  be a cell decomposition of  $S$ . Then, for any fiber  $S_u = \{v : \langle u, v \rangle \in S\}$ ,  $u \in M^n$ , the corresponding family of fibers  $\{D_u : D \in \mathcal{D}, D_u \neq \emptyset\}$  is a cell decomposition of  $S_u$ .*

We will use the following lemma extensively, often without reference. It follows from the full cell decomposition theorem.

**Lemma 2.1.2.** *Let  $X$  be a definable set and let  $f : X \rightarrow (0, \infty)$  be a positive definable function. There exists some definable set  $A \subseteq X$  and some  $\varepsilon > 0$  such that  $\dim A = \dim X$  (in particular  $A$  has nonempty interior in  $X$ ) and, for every  $x \in A$ , it holds that  $f(x) > \varepsilon$ .*

Whenever  $\mathcal{M}$  expands an ordered group, it has definable choice functions i.e. for any definable family of nonempty sets  $\mathcal{S} = \{S_u : u \in \Omega\}$ , there exists a definable function  $f : \Omega \rightarrow \bigcup \mathcal{S}$  such that  $f(u) \in S_u$  for every  $u \in \Omega$ , and moreover  $f(u) = f(v)$  whenever  $u, v \in \Omega$  with  $S_u = S_v$ . As long as  $\mathcal{M}$  has a nonzero 0-definable element,  $f$  can be chosen definable over the same parameters as  $\mathcal{S}$ . For the proof we direct the reader to [17], Chapter 6, Proposition 1.2.

Finally, we also present the Fiber Lemma for o-minimal dimension.

**Lemma 2.1.3** (Fiber Lemma for o-minimal dimension, [17], Chapter 4, Proposition 1.5 and Corollary 1.6). *Let  $X \subseteq M^n$  be a definable set and let  $f : X \rightarrow M^m$  be a definable function. For each  $0 \leq d \leq n$ , let  $S(d) = \{x \in M^m : \dim f^{-1}(x) = d\}$ . Then*

$$\dim X = \max_{0 \leq d \leq n} (\dim S(d) + d).$$

*In particular, the union  $\bigcup \mathcal{S}$  of any infinite definable family  $\mathcal{S}$  of pairwise disjoint sets of dimension  $n$  has dimension strictly greater than  $n$ .*

### 2.1.3 Intersecting families of sets

Recall that a family of sets  $\mathcal{S}$  has the *finite intersection property (FIP)* if  $\bigcap \mathcal{F} \neq \emptyset$  for every finite subfamily  $\mathcal{F} \subseteq \mathcal{S}$ .

We now prove two technical lemmas. The first one will be used in proving the second one, which in turn will be a useful tool in a number of proofs in this thesis.

For the next lemma recall that the dimension of a definable set  $X$  is equal to the dimension of the interpretation of  $X$  in any elementary extension of  $\mathcal{M}$ .

**Lemma 2.1.4.** *Let  $\mathcal{M} \preceq \mathcal{N} = (N, \dots)$ . Let  $X \subseteq M^n$  be a definable set and  $X(\mathcal{N})$  denote the interpretation of  $X$  in  $\mathcal{N}$ . Let  $\dim X = d$ . If  $Y \subseteq N^n$  is an  $\mathcal{N}$ -definable set with  $X \subseteq Y \subseteq X(\mathcal{N})$  then  $\dim Y = d$ .*

*Proof.* Since  $Y \subseteq X(\mathcal{N})$  we have that  $\dim Y \leq \dim X(\mathcal{N}) = d$ . We show, by induction on  $n$ , that from  $X \subseteq Y$  it follows that  $d \leq \dim Y$ . Since otherwise  $X = Y = X(\mathcal{N})$  we may assume that  $X$  is infinite.

Suppose that  $n = 1$ . Since  $X$  is infinite we have that  $Y$  is infinite and so, by o-minimality, it must contain an interval. In particular  $\dim Y \geq 1$ , and so the result follows.

We prove the case  $n > 1$ . By passing to a cell inside  $X$  of maximal dimension if necessary we may assume that  $X$  is a cell. For any  $x \in \pi(Y)$ , let  $Y_x = \{t : \langle x, t \rangle \in Y\}$ .

By induction hypothesis we have that  $\dim \pi(X) \leq \dim \pi(Y)$ . If  $\dim X = \dim \pi(X)$  then it follows that  $d = \dim X = \dim \pi(X) \leq \dim \pi(Y) \leq \dim Y$ , and we are done. The remaining case is that  $\dim X = \dim \pi(X) + 1$ , and  $X$  is of the form  $(f, g)_{\pi(X)}$  for continuous functions  $f, g : \pi(X) \rightarrow M_{\pm\infty}$  with  $f < g$ . Consider the definable set  $Y^{\text{inf}} = \{x \in \pi(Y) : Y_x \text{ is infinite}\}$ . Then  $\pi(X) \subseteq Y^{\text{inf}}$  and by induction hypothesis  $\dim \pi(X) \leq \dim Y^{\text{inf}}$ . By the Fiber Lemma for o-minimal dimension (Lemma 2.1.3)  $\dim Y^{\text{inf}} + 1 \leq \dim Y$ . We conclude that  $d = \dim X = \dim \pi(X) + 1 \leq \dim Y^{\text{inf}} + 1 \leq \dim Y$ , which completes the proof.  $\square$

**Lemma 2.1.5.** *Let  $\{S_u \subseteq M^n : u \in \Omega\}$  be a definable family with the finite intersection property. Then there exists  $\Sigma \subseteq \Omega$  with  $\dim \Sigma = \dim \Omega$  such that  $\bigcap \{S_u : u \in \Sigma\} \neq \emptyset$ .*

*Proof.* Let  $\mathcal{N}$  denote an  $|M|^+$ -saturated elementary extension of  $\mathcal{M}$ . Let  $S_u(\mathcal{N})$ , for any  $u \in \Omega$ , and  $\Omega(\mathcal{N})$  denote the interpretations in  $\mathcal{N}$  of  $S_u$  and  $\Omega$  respectively. By saturation there exists  $x_0 \in N^n$  such that  $x_0 \in S_u(\mathcal{N})$  for every  $u \in \Omega$ . By Lemma 2.1.4, the  $\mathcal{N}$ -definable set  $\Sigma_{x_0}(\mathcal{N}) = \{u \in \Omega(\mathcal{N}) : x_0 \in S_u(\mathcal{N})\}$  has dimension equal to  $\dim \Omega$ . For each  $x \in M^n$ , let  $\Sigma_x = \{u \in \Omega : x \in S_u\}$ . Note that, since  $\dim \Sigma_{x_0}(\mathcal{N}) = \dim \Omega$  and  $\mathcal{M} \preceq \mathcal{N}$ , there must exist some  $x \in M^n$  such that  $\dim \Sigma_x = \dim \Omega$ .  $\square$

#### 2.1.4 Types and tame extensions

Let  $p$  be an  $n$ -type in  $\mathcal{M}$  (over a set  $A \subseteq M$ ). We refer to  $p$  indistinctly as a consistent family of formulas (with parameters from  $A$ ) with  $n$  free variables  $\{\varphi(v) : \varphi \in p\}$  and as a family of definable (over  $A$ ) sets  $\{\varphi(M^n) : \varphi \in p\}$  with the finite intersection property (FIP). For a structure  $\mathcal{N} = (N, \dots)$ , let  $S_n(N)$  denote the set of complete  $n$ -types over  $N$ .



Unless otherwise specified, all types we consider are in  $\mathcal{M}$ , global (over  $M$ ) and complete. By this identification an  $n$ -type is an ultrafilter on  $Def(M^n)$ .

A type  $p$  with object variable  $v$  is ( $A$ -)definable if, for every partitioned formula  $\varphi(u, v)$ , the restriction of  $p$  to  $\varphi(u, v)$ , namely the collection of sets of the form  $\varphi(u, M^{l(v)})$  in  $p$ , is ( $A$ -)definable. In other words if, for every partitioned formula  $\varphi(u, v)$ , the set  $\{u : \varphi(u, M^{l(v)}) \in p\}$  is ( $A$ -)definable.

**Definition 2.1.6.** *A family of sets  $\mathcal{S}$  is downward directed if, for every pair  $S, S' \in \mathcal{S}$ , there exists  $S''$  such that  $S'' \subseteq S \cap S'$ . Moreover, for convenience we ask that  $\mathcal{S}$  does not contain the empty set. Equivalently,  $\mathcal{S}$  is a filter basis. We define the notion of upward directed family analogously.*

**Definition 2.1.7.** *Let  $p$  be an  $n$ -type. A type basis for  $p$  is a partial type  $q \subseteq p$  that is downward directed and generates  $p$ , i.e.  $p = \{X \in Def(M^n) : Y \subseteq X \text{ for some } Y \in q\}$ .*

A uniform (type) basis for  $p$  is a type basis given by a uniform family  $q = \{\varphi(u, M^n) : u \in \Omega\}$ , for some formula  $\varphi(u, v)$  with  $l(v) = n$ .

In o-minimal and weakly o-minimal structures every 1-type  $p$  has a uniform basis. When  $p$  is not realized this basis is given by all the open intervals in  $p$ .

Note that a type with a uniform basis is  $A$ -definable if and only if it has an  $A$ -definable basis.

We now recall the basics of tame extensions and tame pairs.

A tame extension  $\mathcal{N} = (N, <, \dots)$  of  $\mathcal{M}$  is a proper elementary extension of  $\mathcal{M}$  such that  $\{s \in M : s < t\}$  has a supremum in  $M \cup \{-\infty, +\infty\}$  (i.e. the induced cut is definable) for every  $t \in N$ . Note that, by Dedekind completeness, if  $\mathcal{M}$  expands  $(\mathbb{R}, <)$  then every proper elementary extension of  $\mathcal{M}$  is tame.

The following is the canonical example of a tame extension. Suppose that  $\mathcal{M}$  expands an ordered group and let  $\xi$  be a positive element in an elementary extension of  $\mathcal{M}$  which is less than every positive element of  $M$  (i.e.  $\xi$  is infinitesimal with respect to  $M$ ). Then  $\mathcal{M}(\xi) = (M(\xi), \dots)$ , the prime model over  $M \cup \{\xi\}$ , is a tame extension of  $\mathcal{M}$ . Every tuple of elements in  $M(\xi)$  is of the form  $\gamma(\xi)$  for some curve  $\gamma : (0, \varepsilon) \rightarrow N^{l(\gamma(\xi))}$  definable over  $M$ . By o-minimality, it holds that, for every  $(n + m)$ -formula  $\phi(v, u)$  and every  $a \in M^m$ ,

$\mathcal{M}(\xi) \models \phi(\gamma(\xi), a)$  if and only if  $\mathcal{M} \models \phi(\gamma(t), a)$  for all sufficiently small  $t > 0$ . It follows that every type realized in  $\mathcal{M}(\xi)$  is definable. We present the Marker-Steinhorn Theorem, which proves this fact for all tame extensions.

**Theorem 2.1.8** ([31], Theorem 2.1). *Let  $\mathcal{N} = (N, \dots)$  be a tame extension of  $\mathcal{M}$ . For every  $a \in N^n$ , the type  $\text{tp}(a/M)$ , namely the type of  $a$  over  $M$ , is definable. Conversely any definable type is realized in some tame extension of  $\mathcal{M}$ .*

In Appendix 5.A we use the methods developed in Chapter 5 to give a new proof of the Marker-Steinhorn Theorem.

Given a tame extension  $\mathcal{N}$  of  $\mathcal{M}$ , a *tame pair*  $(\mathcal{N}, M)$  is the expansion of  $\mathcal{N}$  by a unary predicate defining  $M$ . Van den Dries and Lewenberg [18] proved that the theory of tame pairs of o-minimal expansions of ordered fields is complete. In other words, suppose that  $\mathcal{M}$  expands an ordered field and let  $T$  denote the theory of  $\mathcal{M}$ . Then, if  $\mathcal{M}_0, \mathcal{N}_0, \mathcal{M}_1$  and  $\mathcal{N}_1$  are models of  $T$  such that  $\mathcal{N}_0$  is a tame extension of  $\mathcal{M}_0$ , and  $\mathcal{N}_1$  is a tame extension of  $\mathcal{M}_1$ , then the tame pairs  $(\mathcal{N}_0, M_0)$  and  $(\mathcal{N}_1, M_1)$  are elementarily equivalent.

## 2.2 Definable topological spaces

Recall that a topological space is  $T_1$  if every singleton is closed,  $T_2$  if it is Hausdorff and  $T_3$  if it is Hausdorff and regular, where regular means that any point  $x$  and closed set  $C \not\ni x$  are separated by neighborhoods, i.e. there exist disjoint open sets  $U, V$  with  $x \in U$  and  $C \subseteq V$ .

We recall the definition of definable topological space given in the introduction.

**Definition 2.2.1.** *A definable topological space is a tuple  $(X, \tau)$ , where  $X \subseteq M^n$  is a definable set<sup>1</sup> and  $\tau$  is a topology on  $X$  such that there exists a definable family of sets  $\mathcal{B}$  that is a basis for  $\tau$ . We call  $\mathcal{B}$  a definable basis for  $\tau$  and say that the topology  $\tau$  is definable.*

Given a definable set  $X$ , the euclidean (Example A.1) and discrete (Example A.2) topologies on  $X$  are definable. We denote these by  $\tau_e$  and  $\tau_s$  respectively, in such a way that the notation remains unambiguous. In particular, whenever  $\mathcal{M}$  expands an ordered group, the

<sup>1</sup>Clearly the fact that  $X$  is definable is redundant

euclidean topology is defined by the definable  $l_\infty$  norm, which we denote  $\|\cdot\|$ . That is, for any  $\langle x_1, \dots, x_n \rangle \in M^n$ , let

$$\|\langle x_1, \dots, x_n \rangle\| = \max\{|x_1|, \dots, |x_n|\}.$$

Let  $(X, \tau)$  be a topological space and  $Z \subseteq X$ . We denote the closure, interior, frontier and boundary of  $Z$  in  $(X, \tau)$  by  $cl_\tau Z$ ,  $int_\tau Z$ ,  $\partial_\tau Z$  and  $bd_\tau Z$  respectively. We also abuse notation by writing  $(Z, \tau)$  to denote the subspace  $(Z, \tau|_Z)$ . Moreover throughout we write  $cl_e$  to mean  $cl_{\tau_e}$ , and more generally write, when used as a prefix or subscript, the letter  $e$  in place of  $\tau_e$  (e.g.  $e$ -open,  $e$ -neighborhood). We adopt analogous conventions with respect to the discrete topology  $\tau_s$ .

The following are some basic facts about definable topological spaces that hold in any first-order structure, regardless of the axiom of o-minimality and the fact that  $\mathcal{M}$  expands a linear order. Familiarity with them will be assumed.

**Proposition 2.2.2.** *Let  $(X, \tau)$  and  $(Y, \mu)$  be definable topological spaces.*

- (a) *If  $\mathcal{B}$  is a definable basis for  $\tau$  then the family  $\mathcal{B}(x) = \{A \in \mathcal{B} : x \in A\}$  is a basis of open neighborhoods of  $x$  that is definable uniformly on  $x \in X$ .*
- (b) *Let  $Z \subseteq X$  be a definable set. Then  $cl_\tau Z$ ,  $int_\tau Z$ ,  $\partial_\tau Z$  and  $bd_\tau Z$  are also definable.*
- (c) *Let  $f : (X, \tau) \rightarrow (Y, \mu)$  be a definable function. The set of points where  $f$  is continuous is definable.*
- (d) *If  $Z \subseteq X$  is a definable set then the subspace  $(Z, \tau)$  is a definable topological space.*
- (e) *The product space  $(X \times Y, \tau \times \mu)$  is a definable topological space.*

For a treatment of definable topological spaces in general first-order model theory we direct the reader to [38].

We end this section with a further key definition.

**Definition 2.2.3.** *A definable topological space  $(X, \tau)$  is affine if it is definably homeomorphic to a set with the euclidean topology.*

A definable topological space  $(X, \tau)$  is locally affine if every point has an affine  $\tau$ -neighborhood.

Affineness is the main topic of research in Section 6.7.1 (one-dimensional definable topological spaces) and in Chapter 7 (spaces of all dimensions).

### 2.2.1 Definable metric spaces

In the spirit of the definition of  $\mathcal{M}$ -norm introduced by Thomas in [48] we present the next definition.

**Definition 2.2.4.** *Suppose that  $\mathcal{M}$  expands an ordered group and let  $X$  be a set. An  $\mathcal{M}$ -metric on  $X$  is a map  $d : X \times X \rightarrow M^{\geq 0}$  that satisfies the metric axioms, i.e. identity of indiscernibles, symmetry and subadditivity.*

The following definition encompasses an important class of definable topological spaces.

**Definition 2.2.5** (Walsberg [49]). *Suppose that  $\mathcal{M}$  expands an ordered group. A definable metric space is a tuple  $(X, d)$  where  $X$  is a definable set and  $d$  is an  $\mathcal{M}$ -metric on  $X$  that is definable.*

Definable metric spaces were defined and studied in the setting of o-minimal expansions of fields by Walsberg [49]. The definition however makes sense in the ordered group case. Whenever we mention definable metric spaces we are always implicitly assuming that  $\mathcal{M}$  expands an ordered group.

Any  $\mathcal{M}$ -metric  $d$  generates a topology in the usual way (and one can easily prove that any such topology is always  $T_3$ ). We denote this topology  $\tau_d$ . In particular any definable metric on a definable set induces a definable topology. By this identification every definable metric space is a definable topological space. Following the conventions set for the euclidean and discrete topologies, we sometimes abuse terminology and write  $d$  in place of  $\tau_d$ , e.g.  $d$ -closed,  $cl_d X \dots$

**Definition 2.2.6.** *A topological space  $(X, \tau)$  is  $\mathcal{M}$ -metrizable if there exists an  $\mathcal{M}$ -metric  $d$  on  $X$  such that  $\tau_d = \tau$  and definably  $\mathcal{M}$ -metrizable if there exists some definable  $\mathcal{M}$ -metric  $d$  on  $X$  such that  $\tau_d = \tau$ .*

**Remark 2.2.7.** From now on we shall simplify our terminology and refer to metrics, rather than  $\mathcal{M}$ -metrics, and similarly to metrizability, rather than  $\mathcal{M}$ -metrizability, without any loss of clarity.

Much like in general topology, there are definable topological spaces which are not definably metrizable. A basic example would be any non-Hausdorff definable topological space, e.g. the Sierpinski Space  $X = \{0, 1\}$ ,  $\tau = \{\emptyset, \{1\}, \{0, 1\}\}$ . A  $T_3$  example is given by the lower limit topology  $\tau_r$  on  $M$  (Example A.3). This is the topology with a definable basis given by all right half-open intervals  $[x, y)$ , for  $x < y$  in  $M$ .

**Proposition 2.2.8.** *The space  $(M, \tau_r)$  is not definably metrizable.*

*Proof.* If  $(M, \tau_r)$  were definably metrizable with definable metric  $d$  then there would exist, for every  $x \in M$ , some  $\varepsilon_x > 0$  such that, for every  $0 < t \leq \varepsilon_x$ ,  $B_d(x, t) \subseteq [x, +\infty)$ , where  $B_d(x, t)$  is the open  $d$ -ball of radius  $t$  and center  $x$ . Let 1 denote some fixed positive element in  $M$  and  $f : X \rightarrow (0, \infty)$  be the definable map given by  $f(x) = \sup\{s \leq 1 : \forall t \in (0, s), B_d(x, t) \subseteq [x, +\infty)\}$ . By o-minimality (Lemma 2.1.2) there exists an interval  $I \subseteq M$  and some  $\varepsilon > 0$  such that  $f(x) > \varepsilon$  for all  $x \in I$ . Hence, by definition of  $f$ , for any distinct  $x, y \in I$  it holds that  $d(x, y) \geq \varepsilon$ , i.e.  $(I, \tau_r)$  is a discrete space, which contradicts the definition of  $\tau_r$ .  $\square$

Suppose that  $\mathcal{M}$  is an expansion of  $(\mathbb{R}, <)$ . The space  $(\mathbb{R}, \tau_r)$ , called the Sorgenfrey Line, is separable but not second countable, and thus it is not even metrizable. On the other hand, if  $\mathcal{M} = (\mathbb{Q}, <)$ , then the lower limit topology is metrizable<sup>2</sup>.

## 2.2.2 Definable connectedness and the frontier dimension inequality

The next two definitions generalize two well-known properties closely related to the euclidean topology.

**Definition 2.2.9.** *A definable topological space  $(X, \tau)$  is definably connected if it is not the union of two disjoint nonempty definable open sets.*

<sup>2</sup>↑See <https://math.stackexchange.com/questions/2331814/existence-of-a-certain-near-metric-map-on-an-ordered-divisible-abelian-group>

A definably connected component of  $(X, \tau)$  is a maximal definably connected definable subset of  $X$ .

A topological characterization of the o-minimality of  $\mathcal{M}$  is given by saying that  $(M, \tau_e)$  is definably connected and every definable subset of  $M$  can be partitioned into finitely many definably connected euclidean spaces.

It is not clear whether or not every point in a definable topological space is contained in a (definable) definably connected component. We give a positive answer to this question for one-dimensional Hausdorff regular spaces in Section 6.9.

The following definition is not topological in flavour, but it will be an important tool in formulating and proving statements about definable topological spaces.

**Definition 2.2.10.** *We say a definable topological space  $(X, \tau)$  has the frontier dimension inequality (f.d.i.) if, for every definable set  $Y \subseteq X$ ,  $\dim \partial_\tau Y < \dim Y$ .*

Recall that the euclidean topology has the frontier dimension inequality. Walsberg proved ([49], Lemma 7.15) that every definable metric space satisfies the frontier dimension inequality.

By an inductive argument on dimension, it is easy to show that in any space with the frontier dimension inequality every definable set is a boolean combination of definable open sets (i.e. Property (A) in [38], Section 2).

### 2.2.3 Definable curves

Definable curves play a crucial role in the study of definable topological spaces, often taking the role that sequences have in general topology.

**Definition 2.2.11.** *Let  $(X, \tau)$  be a definable topological space. A curve in  $X$  is a map  $\gamma : (a, b) \rightarrow X$ , where  $a, b \in M_{\pm\infty}$ ,  $a < b$ .*

*We say that  $\gamma$  converges in the  $\tau$ -topology (in  $(X, \tau)$ , or  $\tau$ -converges) as  $t$  tends to  $a$  to a point  $x \in X$  if, for every  $\tau$ -neighborhood  $A$  of  $x$ , there exists some  $a < t_A < b$  such that  $\gamma(t) \in A$  for all  $a < t < t_A$ . In this case we write  $x = \tau\text{-}\lim_{t \rightarrow a} \gamma(t)$ . Convergence as  $t$  tends to  $b$  is defined analogously. When we say that  $\gamma$   $\tau$ -converges to  $x \in X$  we are implicitly fixing*

an extreme point  $c \in \{a, b\}$  and saying that  $\gamma$   $\tau$ -converges to  $x$  as  $t$  tends to  $c$ . We say that  $\gamma$  is  $\tau$ -convergent if it  $\tau$ -converges to some point.

**Remark 2.2.12.** We adopt some further conventions regarding definable curves. Let  $\gamma : (a, b) \rightarrow M^n$  be a definable map. By o-minimality, for any definable set  $X \subseteq M^n$  there exists  $a' > a$  such that either  $\gamma[(a, a')] \subseteq X$  or  $\gamma[(a, a')] \subseteq M^n \setminus X$ , and the analogous statement holds for  $b$ .

Most of the time we only care about the behaviour of  $\gamma$  near one of the extremes  $a$  or  $b$ . Hence if there is (say)  $a' > a$  such that  $\gamma[(a, a')] \subseteq X$  we may treat  $\gamma$  as a curve in  $X$  by implicitly identifying it with its restriction  $\gamma|_{(a, a')}$ .

Similarly we say that  $\gamma$  is constant or injective (or some other property) if it has this property when restricting its domain to an appropriate interval as above. By o-minimality every definable curve  $\gamma : (a, b) \rightarrow M^n$  can be assumed to be either constant or injective (strictly monotonic if  $n = 1$ ) and continuous (with respect to the euclidean topology).

The following is a weak version of well-known property of euclidean spaces in o-minimal expansions of ordered groups. We drop the condition that the curves be continuous.

**Definition 2.2.13.** A definable topological space  $(X, \tau)$  has definable curve selection if, for every definable set  $Z \subseteq X$  and any  $x \in \text{cl}_\tau Z$  there exists a definable curve  $\gamma$  in  $Z$   $\tau$ -converging to  $x$ .

It is easy to see that, whenever the underlying structure  $\mathcal{M}$  has definable choice, any definable metric space has definable curve selection. In Chapter 4 (Lemma 4.5.4) we prove that, whenever  $\mathcal{M}$  expands an ordered field, any definable topological space has definable curve selection.

Definable curve selection allows the characterization of continuity in terms of convergence of definable curves as follows.

**Proposition 2.2.14.** Let  $(X, \tau)$  and  $(Y, \mu)$  be definable topological spaces, where  $(X, \tau)$  has definable curve selection. Let  $f : (X, \tau) \rightarrow (Y, \mu)$  be a definable map. Then  $f$  is continuous at  $x \in X$  if and only if, for every definable curve  $\gamma : (a, b) \rightarrow X$  and  $c \in \{a, b\}$ , if  $\tau\text{-}\lim_{t \rightarrow c} \gamma(t) = x$  then  $\mu\text{-}\lim_{t \rightarrow c} (f \circ \gamma)(t) = f(x)$ .

*Proof.* Let  $\gamma : (a, b) \rightarrow X$  be a definable curve  $\tau$ -converging to  $x \in X$  as (say)  $t \rightarrow a$ . If  $f$  is continuous at  $x$  then, for every  $\mu$ -neighborhood  $A$  of  $f(x)$ ,  $f^{-1}(A)$  is a  $\tau$ -neighborhood of  $x$ , so there is  $a < t_A < b$  such that  $\gamma[(a, t_A)] \subseteq f^{-1}(A)$ . So, for every  $a < t < t_A$ ,  $(f \circ \gamma)(t) \in A$ , meaning that  $f \circ \gamma$   $\mu$ -converges to  $f(x)$ .

Conversely, suppose  $f$  is not continuous at  $x \in X$ . Thus there exists some definable  $\mu$ -neighborhood  $A$  of  $f(x)$  such that  $f^{-1}(A)$  is not a  $\tau$ -neighborhood of  $x$ , i.e.  $x \in cl_\tau(X \setminus f^{-1}(A))$ . By definable curve selection there exists a definable curve  $\gamma$  in  $X \setminus f^{-1}(A)$   $\tau$ -converging to  $x$ . However, by definition,  $f \circ \gamma$  lies in  $Y \setminus A$ , and so does not  $\mu$ -converge to  $f(x)$ .  $\square$

## 2.2.4 Push-forwards

**Definition 2.2.15.** Let  $(X, \tau)$  be a definable topological space with definable basis  $\mathcal{B}$  and let  $f : X \rightarrow M^m$  be an injective definable map. We define the push-forward of  $(X, \tau)$  by  $f$  to be the definable topological space  $(f(X), f(\tau))$  where  $f(\tau)$  is the topology on  $f(X)$  with definable basis  $\{f(A) : A \in \mathcal{B}\}$ . Thus  $f(\tau)$  is the topology satisfying that  $f : (X, \tau) \rightarrow (f(X), f(\tau))$  is a homeomorphism.

We will make use of the next remark extensively. It allows us to make the assumption, whenever  $\mathcal{M}$  expands an ordered field, that any definable topological space of dimension  $n > 0$  is, up to definable homeomorphism, a bounded subset of  $M^n$ .

**Remark 2.2.16.** Let  $X$  be a definable set with  $\dim(X) = m > 0$ . If  $\mathcal{M}$  expands an ordered group and  $X$  is bounded there exists a definable injection  $f : X \rightarrow M^m$ . In particular, if  $\tau$  is a definable topology on  $X$  then, by passing to the push-forward of  $(X, \tau)$  by  $f$  if necessary, one may always assume, up to definable homeomorphism, that  $X \subseteq M^m$ . If moreover  $\mathcal{M}$  expands an ordered field then the injection  $f$  exists without the assumption that  $X$  is bounded, and with the added condition that  $f(X) \subseteq (0, 1)^m$ . In particular one may always assume, up to definable homeomorphism, that  $X \subseteq (0, 1)^m$ .

The existence of one such injection follows from noting first that, whenever  $\mathcal{M}$  expands an ordered field, the map given coordinate-wise by  $x_i \mapsto \frac{2x_i - 1}{(2x_i - 1)^2 - 1}$  gives a definable homeomorphism from  $(0, 1)^n$  to  $M^n$  for every  $n$ . Moreover, if  $\mathcal{M}$  expands an ordered group and  $X$  is



a bounded definable set with  $\dim X = m > 0$ , there exists a definable injection  $f : X \rightarrow M^m$  as follows.

Let  $\mathcal{D}$  be a finite partition of  $X$  into cells. By o-minimality each cell in  $C \in \mathcal{D}$  is in bijection, under an appropriate projection  $\pi_C$ , with a subset of  $M^m$ . Since  $X$  is bounded then so are the sets  $\pi_C(C)$  for  $C \in \mathcal{D}$ . Since  $\mathcal{M}$  is an expansion of a group, we can find appropriate translations of these sets such that they do not intersect each other. The union of the image of these translations is then in definable bijection with  $X$ .

### 2.2.5 $e$ -Accumulation sets

The following definition will be used to study one-dimensional definable topological spaces in Chapter 6. For example, it plays a crucial role in Lemma 6.2.11, which describes bases of neighborhoods of points in  $T_1$  definable topological spaces  $(X, \tau)$  where  $X \subseteq M$ .

**Definition 2.2.17.** *Let  $(X, \tau)$ ,  $X \subseteq M^n$ , be a definable topological space. Let  $x \in X$  and  $\mathcal{B}(x)$  be a basis of  $\tau$ -neighborhoods of  $x$ . We define the  $e$ -accumulation set of  $x$  in  $(X, \tau)$ , namely  $E_x^{(X, \tau)}$ , to be:*

$$E_x^{(X, \tau)} := \bigcap_{A \in \mathcal{B}(x)} \{y \in (M_{\pm\infty})^n : B \setminus \{y\} \cap A \neq \emptyset, \forall B \in \tau_e, y \in B\},$$

where  $\tau_e$  refers to the euclidean topology in  $(M_{\pm\infty})^n$ . So  $E_x^{(X, \tau)}$  is the intersection of the set of accumulation points (in the euclidean topology understood in  $(M_{\pm\infty})^n$ ) of every  $\tau$ -neighborhood of  $x$ .

If  $(X, \tau)$  is  $T_1$  and  $x \neq y$  then  $y \in E_x^{(X, \tau)}$  is equivalent to stating that, for every  $\tau$ -neighborhood  $A$  of  $x$  and every  $e$ -neighborhood  $B$  of  $y$ ,  $A \cap B \neq \emptyset$ .

The definition of  $E_x^{(X, \tau)}$  is clearly independent of the choice of basis of neighborhoods. Note that, if an element  $x \in X$  is isolated, then it satisfies  $E_x^{(X, \tau)} = \emptyset$ . We show in Chapter 6 (Lemma 6.2.11) that the converse is not necessarily true. For any point  $x$  in a euclidean space  $(M^n, \tau_e)$  it holds that  $E_x^{(M^n, \tau_e)} = \{x\}$ .

Generally, since there will be no room for confusion, once a definable topological space  $(X, \tau)$  is fixed then for any  $x \in X$  we will write  $E_x$  in place of  $E_x^{(X, \tau)}$ , and will only resort to

the latter when we also intend to address the  $e$ -accumulation set  $E_x^{(Y,\tau)}$  for some definable subspace  $Y$  containing  $x$ .

The following proposition states facts regarding  $e$ -accumulation sets that follow immediately from the definition. Note that in it the euclidean topology is understood in  $M_{\pm\infty}$ .

**Proposition 2.2.18.** *Let  $(X, \tau)$  be a definable topological space.*

- (a)  $E_x$  is  $e$ -closed and  $E_x \subseteq cl_e X$  for every  $x \in X$ .
- (b) The relation  $\{\langle x, y \rangle : y \in E_x\} \subseteq X \times M_{\pm\infty}^m$  is definable.

We now use Lemma 2.1.5 to prove a bound on the dimension of  $e$ -accumulation sets.

**Definition 2.2.19.** *Let  $(X, \tau)$  be a definable topological space. Let  $x \in X$  and let  $\mathcal{B}(x)$  be a definable basis of neighborhoods of  $x$  in  $(X, \tau)$ . The local dimension of  $(X, \tau)$  at  $x$  is*

$$\dim_x(X, \tau) = \min\{\dim A : A \in \mathcal{B}(x)\}.$$

Clearly the definition of local dimension does not depend on the choice of basis of neighborhoods. This definition generalizes the definition of local dimension of a definable metric space at a point that was introduced by Walsberg in [49].

**Lemma 2.2.20.** *Let  $(X, \tau)$  be a  $T_1$  definable topological space. For any  $x \in X$ ,  $\dim(E_x) < \dim_x(X, \tau)$ . In particular when  $\dim X \leq 1$  then the set  $E_x$  is finite for every  $x \in X$ .*

*Proof.* Let  $\{A_u : u \in \Omega\}$  be a definable basis of  $\tau$ -neighborhoods of  $x$ . If  $\dim_x(X, \tau) = 0$ , then, by definition of  $E_x$ , we have that  $E_x = \emptyset$ . From now on we assume that  $\dim_x(X, \tau) > 0$ .

If  $\dim E_x \leq 0$  then the proof is complete. Suppose that  $\dim E_x > 0$ , and in particular that  $\dim E_x = \dim E_x \setminus \{x\}$ . For any  $y \in E_x \setminus \{x\}$ , let  $\Omega_y = \{u \in \Omega : y \notin A_u\}$ . Since  $X$  is  $T_1$  the sets  $\Omega_y$  are nonempty, and in fact the definable family  $\{\Omega_y : y \in E_x \setminus \{x\}\}$  has the finite intersection property.

Applying Lemma 2.1.5 there exists a definable set  $Y \subseteq E_x \setminus \{x\}$  with  $\dim Y = \dim E_x$  and  $u \in \Omega$  such that  $A_u \cap Y = \emptyset$ . By shrinking  $A_u$  if necessary we may assume that  $\dim A_u = \dim_x(X, \tau)$ . Note however that, by definition of  $E_x$ ,  $Y \subseteq cl_e A_u$ , and so  $Y \subseteq \partial_e A_u$ .

In particular  $\dim E_x = \dim Y \leq \dim \partial_e A_u$ . However by o-minimality  $\dim \partial_e A_u < \dim A_u$ , so  $\dim E_x < \dim A_u = \dim_x(X, \tau)$ .  $\square$

### 3. DEFINABLE SEPARABILITY

#### Introduction

In this chapter we introduce and characterize a notion of definable separability. Our main result is Theorem 3.2.6, which states that, whenever  $\mathcal{M}$  expands  $(\mathbb{R}, <)$ , definable separability, classical separability, and the countable chain condition are equivalent.

Our notion of definable separability is inspired by but different from the one introduced by Walsberg for definable metric spaces [49]. We show that, when restricted to definable metric spaces, both notions are equivalent, and argue that Walsberg's definition is not suitable for general definable topological spaces.

In this chapter we make extensive use of the Fiber Lemma for o-minimal dimension (Lemma 2.1.3).

#### 3.1 Definitions and basic results

The following is the main definition of this chapter.

**Definition 3.1.1.** *A definable topological space  $(X, \tau)$  is definably separable if there exists no infinite definable family of open pairwise disjoint sets in  $\tau$ .*

The reader will note the similarity between Definition 3.1.1 and the countable chain condition (**ccc**, or Suslin's condition) for topological spaces. That is, a topological space has the **ccc** if it does not contain an uncountable family of pairwise disjoint open sets. In general every separable topological space has the **ccc**, but the converse is not true. We prove in Theorem 3.2.6 that, for spaces definable in an o-minimal expansion of  $(\mathbb{R}, <)$ , having the **ccc** and being separable are equivalent.

**Remark 3.1.2.** Every euclidean space  $X$  is definably separable. If  $X$  is  $e$ -open then this follows easily from the Fiber Lemma for o-minimal dimension (Lemma 2.1.3). In general it follows from cell decomposition. That is, if  $X$  fails to be definably separable, then there must be a cell that is also not definably separable. However every cell is homeomorphic through a projection to an open cell.

Additionally note that, again by the use of cell decomposition, every definable euclidean space in an o-minimal expansion of  $(\mathbb{R}, <)$  is separable, since every cell in a structure expanding the reals is separable.

Walsberg [49] introduced the following definition of definable separability for definable metric spaces.

**Definition 3.1.3** (Walsberg [49]). *A definable metric space  $(X, d)$  is definably separable if it does not contain an infinite definable discrete subspace.*

Our Definition 3.1.1 was inspired by Definition 3.1.3. The differences between them are due to the fact that there exist (definably non-metrizable) definable topological spaces in  $(\mathbb{R}, <, +, \cdot)$  that are separable but also contain an infinite definable discrete subspace, for such examples see Example A.9 and the definable Moore Plane (Example A.12) in Appendix A.

We explain the relationship between both our definition and the generalization of Walsberg's to all definable topological spaces in the next lemma and proposition, showing in particular that they are equivalent for definable metric spaces. We make use of the following definition.

**Definition 3.1.4.** *A definable topological space  $(X, \tau)$  is hereditarily definably separable if every definable subspace of  $(X, \tau)$  is definably separable.*

Clearly every hereditarily definably separable space is in particular definably separable. We observe how, as long as  $\mathcal{M}$  has definable choice, the generalization of Definition 3.1.3 to all definable topological spaces is equivalent to hereditary definable separability.

**Lemma 3.1.5.** *Suppose that  $\mathcal{M}$  has definable choice. A definable topological space  $(X, \tau)$  is hereditarily definably separable if and only if it does not contain an infinite definable discrete subspace.*

*Proof.* Any infinite discrete space is clearly not definably separable and so the “only if” implication follows.

For the “if” implication let  $Y \subseteq X$  be a definable subset such that  $(Y, \tau)$  is not definably separable. Let  $\mathcal{A} = \{A_u : u \in \Omega\}$  be an infinite pairwise disjoint family of open sets in

$(Y, \tau)$ . Using definable choice let  $f : \Omega \rightarrow \cup \mathcal{A}$  be a definable map such that  $f(u) \in A_u$  for every  $u \in \Omega$ , where  $f(u) = f(v)$  whenever  $A_u = A_v$ . Then the image  $(f(\Omega), \tau)$  is an infinite definable discrete subspace.  $\square$

Recall that definable metric spaces are understood implicitly in the setting where  $\mathcal{M}$  expands an ordered group, and so in particular  $\mathcal{M}$  has definable choice. Hence, by Lemma 3.1.5, Walsberg's notion of definable separability for definable metric spaces is equivalent to Definition 3.1.4.

We further characterize definable separability for definable metric spaces as follows.

**Proposition 3.1.6.** *Let  $(X, d)$  be a definable metric space. The following are equivalent.*

- (1)  $(X, d)$  is definably separable.
- (2)  $(X, d)$  is hereditarily definably separable.
- (3)  $(X, d)$  does not contain an infinite definable discrete subspace, i.e. it is definably separable in the sense of Walsberg [49].

*Proof.* The equivalence (2)  $\Leftrightarrow$  (3) is given by Lemma 3.1.5. The implication (2)  $\Rightarrow$  (1) is trivial. We prove (1)  $\Rightarrow$  (3).

Let  $(X, d)$  be a definable metric space and  $Y$  a definable discrete subspace. By definable choice one may select definably, for each  $x \in Y$ , some  $\varepsilon_x > 0$  such that, for every  $y \in Y \setminus \{x\}$ ,  $2\varepsilon_x < d(x, y)$ . We observe that the infinite definable family of open  $d$ -balls  $\{B_d(x, \varepsilon_x) : x \in Y\}$  is pairwise disjoint, and so  $(X, d)$  is not definably separable.

Towards a contradiction suppose that there exists  $x, y \in Y$  and some  $z \in B_d(x, \varepsilon_x) \cap B_d(y, \varepsilon_y)$ . Then, by the triangle inequality,

$$d(x, y) \leq d(x, z) + d(z, y) \leq \varepsilon_x + \varepsilon_y \leq 2 \max\{\varepsilon_x, \varepsilon_y\}.$$

Without loss of generality suppose that  $\varepsilon_x = \max\{\varepsilon_x, \varepsilon_y\}$ . Then this contradicts the fact that  $2\varepsilon_x < d(x, y)$ .  $\square$

Note that the proof of Proposition 3.1.6 (and of Lemma 3.1.5) relies solely on definable choice and not on the fact that the structure  $\mathcal{M}$  is o-minimal.

In general topology every separable metric space is second countable and thus hereditarily separable. On the other hand, in parallelism with the o-minimal definable setting, there exist (non-metrizable) topological spaces, such as the Moore Plane and Sorgenfrey Plane, that are separable but not hereditarily separable.

### 3.2 Main result

In this section we will make use of the first paragraph of the following lemma. The second paragraph will be used in Chapter 7.

**Lemma 3.2.1.** *Let  $\mathcal{S} \subseteq \mathcal{P}(R^n)$  be a definable upward directed family of sets. Then  $\dim \cup \mathcal{S} = \max\{\dim S : S \in \mathcal{S}\}$ .*

*Suppose that  $\mathcal{S} \subseteq \mathcal{P}(X)$  for some definable set  $X \subseteq R^n$ . If  $S$  has empty interior in  $X$  for every  $S \in \mathcal{S}$  then  $\cup \mathcal{S}$  has empty interior in  $X$ .*

*Proof.* Suppose that  $\mathcal{S} = \{S_u : u \in \Omega\}$  and set  $Y := \cup \mathcal{S}$ . We may clearly assume that  $Y \neq \emptyset$ . For every  $x \in Y$  the set  $\Omega_x = \{u \in \Omega : x \in S_u\}$  is definable. The definable family  $\{\Omega_x : x \in Y\}$  has the finite intersection property so, by Lemma 2.1.5, there exists  $Y' \subseteq Y$  with  $\dim Y' = \dim Y$  such that  $\cap\{\Omega_x : x \in Y'\} \neq \emptyset$ . Hence there is  $u \in \Omega$  such that  $Y' \subseteq S_u$ , so  $\dim Y = \dim Y' \leq \dim S_u \leq \max\{\dim S_u : u \in \Omega\} \leq \dim Y$ . This proves the first paragraph of the lemma.

Now suppose that  $\mathcal{S} \subseteq \mathcal{P}(X)$  for some definable set  $X \subseteq R^n$ . Suppose that  $Y$  has nonempty interior  $Z$  in  $X$ . Clearly the definable family  $\{S_u \cap Z : u \in \Omega\}$  is upward directed and satisfies that  $\cup_{u \in \Omega} (S_u \cap Z) = Z$ . By the statement in the first paragraph of the lemma there exists  $u \in \Omega$  such that  $\dim(S_u \cap Z) = \dim Z$ . By o-minimality it follows that  $S_u \cap Z$  has interior in  $Z$ , and thus in  $X$ . □

The following ad hoc lemma is trivial whenever  $\mathcal{M}$  has elimination of imaginaries (e.g. expands an ordered group).

**Lemma 3.2.2.** *Suppose that  $\mathcal{M}$  expands  $(\mathbb{R}, <)$ . Any infinite definable family of sets is uncountable.*

*Proof.* If  $\{S_u : u \in \Omega\}$  is a definable family of sets then the relation  $u \sim v \Leftrightarrow S_u = S_v$  is a definable equivalence relation. Hence it suffices to prove that any infinite quotient space by a definable equivalence relation is uncountable. We use the fact that any infinite definable subset in an o-minimal expansion of  $(\mathbb{R}, <)$  is uncountable.

Let  $\Omega \subseteq M^n$  be a definable set and  $\sim$  denote a definable equivalence relation on  $\Omega$  with an infinite quotient space  $\Omega/\sim$ . For any  $u \in \Omega$ , we denote the corresponding equivalence class by  $[u]_\sim$ . We proceed by induction on  $n$ .

Suppose that  $n = 1$ . For any  $u \in \Omega$  let  $f(u) = \inf[u]_\sim$ . Note that, by o-minimality, for every  $t \in M \cup \{-\infty\}$  there can be at most two non-equivalent elements  $u, v \in \Omega$  with  $f(u) = f(v) = t$ . Since  $\Omega/\sim$  is infinite it follows that the image of  $f$  is infinite. Since  $f$  is definable the result follows.

Now suppose that  $n > 1$ . For any  $x \in \pi(\Omega)$ , where  $\pi$  denotes the projection  $M^n \rightarrow M^{n-1}$ , let  $\Omega_x$  be the fiber  $\{t \in M : \langle x, t \rangle \in \Omega\}$  and  $\sim_x$  the equivalence relation on  $\Omega_x$  given by  $s \sim_x t \Leftrightarrow \langle x, t \rangle \sim \langle x, s \rangle$ . If there exists some  $x \in \pi(\Omega)$  such that  $\Omega_x/\sim_x$  is infinite then, by the base case  $n = 1$ ,  $\Omega_x/\sim_x$  is uncountable, and it follows that  $\Omega/\sim$  is uncountable.

Onwards suppose that, for every  $x \in \pi(\Omega)$ ,  $\Omega_x/\sim_x$  is finite. We first observe that there is a uniform bound on the size of  $\Omega_x/\sim_x$ .

Consider, for every  $x \in \pi(\Omega)$ , the set  $B_x = \{\inf[t]_{\sim_x} : t \in \Omega_x\}$ . For every  $x \in \pi(\Omega)$ , since  $\Omega_x/\sim_x$  is finite, the set  $B_x$  is finite. By uniform finiteness there exists some  $m$  such that  $|B_x| \leq m$  for every  $x \in \pi(\Omega)$ . Now note that, by o-minimality, if  $|B_x| = i$  then the size of  $\Omega_x/\sim_x$  is at most  $2i$ . In particular, for every  $x \in \Omega_x$ , the size of the quotient space  $\Omega_x/\sim_x$  is at most  $2m$ .

For every  $1 \leq i \leq 2m$ , we define  $\Omega(i)$  inductively as follows. Let  $\Omega(1)$  be the set of points  $\langle x, t \rangle \in \Omega$  such that, for every  $s \in \Omega_x$ , there exists some  $t' \in \Omega_x$  with  $\langle x, t \rangle \sim \langle x, t' \rangle$  and  $t' \leq s$ . Since every quotient space  $\Omega_x/\sim_x$  is finite this set is well defined. Now, for  $i > 1$ , let  $\Omega(i)' = \Omega \setminus \cup_{0 \leq j < i} \Omega(j)$ , and let  $\Omega(i)$  be the set of points  $\langle x, t \rangle \in \Omega(i)'$  such that, for every  $s$  in the fiber  $\Omega(i)'_x$ , there exists some  $t' \in \Omega_x$  with  $\langle x, t \rangle \sim \langle x, t' \rangle$  and  $t' \leq s$ .

Note that the sets  $\Omega(i)$ , for  $1 \leq i \leq 2m$ , are definable and partition  $\Omega$ . We fix  $i$  satisfying that  $\Omega(i)/\sim$  is infinite. We show that  $\Omega(i)/\sim$  is uncountable.



For every  $x \in \pi(\Omega(i))$  let  $\Omega(i)_x = \{t \in M : \langle x, t \rangle \in \Omega(i)\}$ . Note that, by construction of  $\Omega(i)$ , for every  $s, t \in \Omega(i)_x$  it holds that  $s \sim_x t$ . Consider the equivalence relation  $\sim_{pr}$  on  $\pi(\Omega(i))$  given by  $x \sim_{pr} y$  if and only if, for every  $t \in \Omega(i)_x$  and  $s \in \Omega(i)_y$ , it holds that  $\langle x, t \rangle \sim \langle y, s \rangle$ . Note that the projection  $\pi : \Omega(i) \rightarrow \pi(\Omega(i))$  induces, through the map  $[\langle x, t \rangle]_{\sim} \mapsto [x]_{\sim_{pr}}$ , a bijection between  $\Omega(i)/\sim$  and  $\pi(\Omega(i))/\sim_{pr}$ . By induction hypothesis the latter space is uncountable, and the result follows.  $\square$

From Lemma 3.2.2 we will derive that any space definable in an o-minimal expansion of  $(\mathbb{R}, <)$  that is not definably separable also fails to have the **ccc**.

The following lemmas are part of the proof that, in an o-minimal expansion of  $(\mathbb{R}, <)$ , any definably separable definable topological space is separable.

**Lemma 3.2.3.** *Let  $(X, \tau)$  be a definably separable definable topological space. Let  $I \subseteq M$  be an open interval and  $\{A_t : t \in I\}$  be a definable family of  $\tau$ -open sets. Suppose that, for every  $t \in I$ ,  $\dim A_t \leq n$ . Then there exists a definable  $\tau$ -open set  $B \subseteq \cup_{t \in I} A_t$  such that  $\dim B \leq n$  and  $\cup_{t \in I} A_t \subseteq cl_{\tau} B$ . In other words, the union of the  $A_t$ 's has a definable  $\tau$ -open dense subset of dimension less than or equal to  $n$ .*

*Moreover,  $B$  can be chosen definable over the same parameters as the family  $\{A_t : t \in I\}$  and the topology  $\tau$ .*

*Proof.* Let  $\mathcal{A} = \{A_t : t \in I\}$  and  $A = \cup_{t \in I} A_t$ . For each  $x \in A$ , let  $I_x = \{t \in I : x \in A_t\}$ . We first show that we may assume that, for any  $x \in A$ ,  $\inf I_x \notin I_x$ . We then show that, under this assumption, it suffices to take  $B = A$ . The fact that  $B$  is definable over the same parameters as  $\mathcal{A}$  and  $\tau$  follows by keeping track of parameters.

For each  $t \in I$ , let  $A'_t = \cup\{A_s : s < t, s \in I\} \cap A_t$ . Let  $\mathcal{A}' = \{A'_t : t \in I\}$ . Note that  $\mathcal{A}'$  is a definable family of  $\tau$ -open sets. For every  $x \in \cup \mathcal{A}'$  let  $I'_x = \{t \in I : x \in A'_t\}$ . Note that, for every  $x \in \cup \mathcal{A}'$ ,  $I'_x = I_x \setminus \{\inf I_x\}$ . In particular, if  $I_x$  is a left-closed interval then  $I'_x$  is a left-open interval. We can then repeat this process, and define, for every  $t \in I$ ,  $A''_t = \cup\{A'_s : s < t, s \in I\} \cap A'_t$ , and a definable family of  $\tau$ -open sets  $\mathcal{A}'' = \{A''_t : t \in I\}$ . Now, by uniform finiteness, there exists some  $m$  such that, for every  $x \in A$ ,  $I_x$  has at most  $m$  definably connected components. Consequently, after repeating the process  $m$  times, we will

reach a definable family of  $\tau$ -open sets  $\mathcal{A}^{(m)} = \{A_t^{(m)} : t \in I\}$  with the following property. For every  $x \in \cup \mathcal{A}^{(m)}$ , if  $I_x^{(m)} = \{t \in I : x \in A_t^{(m)}\}$ , then  $\inf I_x^{(m)} \notin I_x^{(m)}$ .

We proceed by showing that it suffices to prove the lemma for  $\mathcal{A}'$  in place of  $\mathcal{A}$ . Then by an inductive argument the same holds for  $\mathcal{A}^{(m)}$ . Consequently, by passing to  $\mathcal{A}^{(m)}$  if necessary, we may assume that  $\inf I_x \notin I_x$  for every  $x \in A$ .

Recall that, for every  $t \in I$ ,  $A'_t = \cup\{A_s : s < t, s \in I\} \cap A_t$ . Note that, for any two distinct  $s, t \in I$ ,  $(A_s \setminus A'_s) \cap (A_t \setminus A'_t) = \emptyset$ . By definable separability of  $(X, \tau)$  it follows that the set  $\text{int}_\tau(A_t \setminus A'_t)$  is nonempty for only finitely many values of  $t \in I$ . In particular  $\dim(\cup_{t \in I} \text{int}_\tau(A_t \setminus A'_t)) \leq n$ . Moreover, because the sets in  $\mathcal{A}$  are  $\tau$ -open, note that

$$A \subseteq \text{cl}_\tau(\cup_{t \in I} A'_t) \cup \cup_{t \in I} \text{int}_\tau(A_t \setminus A'_t).$$

Let  $A' = \cup_{t \in I} A'_t$ . Suppose that the lemma holds for the family  $\mathcal{A}' = \{A'_t : t \in I\}$  in place of  $\mathcal{A}$ . That is, there exists a definable  $\tau$ -open set  $B' \subseteq A'$  such that  $\dim B' \leq n$  and  $A' \subseteq \text{cl}_\tau B'$ . Then it suffices to consider  $B = B' \cup \cup_{t \in I} \text{int}_\tau(A_t \setminus A'_t)$  to complete the proof.

Hence we conclude that we may assume that  $\mathcal{A}$  satisfies that, for every  $x \in A$ ,  $\inf I_x \notin I_x$ . We complete the proof by showing that, with this assumption,  $\dim A \leq n$ .

Let  $I = (a, b)$ . For any  $t \in [a, b)$ , let

$$C_t = \{x \in A : t = \inf I_x\}.$$

Clearly  $A = \cup\{C_t : t \in [a, b)\}$ . We show that  $\dim C_t \leq n$  for every  $t \in [a, b)$ , with equality holding for only finitely many values of  $t$ . By the Fiber Lemma for o-minimal dimension (Lemma 2.1.3) we conclude that  $\dim A \leq n$ .

Observe that  $C_a = \cup_{a < s < b} \cap_{a < t' < s} A_{t'}$ . Moreover, since  $\inf I_x \notin I_x$  for every  $x \in A$ , we have that, for every  $t \in (a, b)$ ,

$$C_t = \left( \bigcup_{t < s < b} \bigcap_{t < t' < s} A_{t'} \right) \setminus \bigcup_{a < s' \leq t} A_{s'} = \bigcup_{t < s < b} \left( \bigcap_{t < t' < s} A_{t'} \setminus \bigcup_{a < s' \leq t} A_{s'} \right).$$

In particular each set  $C_t$  is a union of nested sets each of which has dimension at most  $n$ . By Lemma 3.2.1, it follows that  $\dim C_t \leq n$ .

Now, towards a contradiction suppose that there exist infinitely many values  $t \in [a, b)$  such that  $\dim C_t = n$ . Then by o-minimality there exists an interval  $J \subseteq I$  that contains only such values. For each  $a < t < s < b$  let

$$C(t, s) = \bigcap_{t < t' < s} A_{t'} \setminus \bigcup_{a < s' \leq t} A_{s'},$$

in which case  $C_t = \bigcup_{t < s < b} C(t, s)$ . Note that, for distinct  $t_0, t_1 \in I$ , and any choice of  $t_0 < s_0 < b$  and  $t_1 < s_1 < b$ , it holds that the sets  $C(t_0, s_0)$  and  $C(t_1, s_1)$  are disjoint.

Applying Lemma 3.2.1, for every  $t \in J$  let  $f(t)$  be the supremum of all the values  $s > t$  in  $J$  such that  $\dim C(t, s) = n$ . The function  $f$  is clearly definable. It moreover satisfies that  $f(t) > t$  for every  $t \in J$ . By o-minimality  $f$  is piecewise continuous, and so we may find an interval  $J' \subseteq J$  such that  $f(t) > J'$  for every  $t \in J'$ .

Finally, let us fix some  $t_0 \in J'$ . Note that, for every  $t_1, s_1$  in  $J'$  with  $t_1 < t_0 < s_1$ , it holds that  $C(t_1, s_1) \subseteq A_{t_0}$ . Moreover, by definition of  $f$ ,  $\dim C(t_1, s_1) = n$ . Also recall that, for any  $t_2 < t_0$  in  $J' \setminus \{t_1\}$ , it holds that  $C(t_1, s_1) \cap C(t_2, s_1) = \emptyset$ . Hence

$$\bigcup_{t < t_0; t \in J} C(t, s_1) \subseteq A_{t_0},$$

where  $\bigcup_{t < t_0; t \in J} C(t, s_1)$  is a union of infinitely many pairwise disjoint sets of dimension  $n$ . By the Fiber Lemma for o-minimal dimension (Lemma 2.1.3) we derive that  $\dim A_{t_0} > n$ , contradiction.  $\square$

Note that, if  $(X, \tau)$  satisfies the frontier dimension inequality (Definition 2.2.10), then in Lemma 3.2.3 we may always take  $B = \bigcup_{t \in I} A_t$ . The analogous remains true for the following generalization of the lemma.

**Lemma 3.2.4.** *Let  $(X, \tau)$  be a definably separable definable topological space. Let  $\Omega \subseteq M^m$  be a definable set and  $\{A_u : u \in \Omega\}$  be a definable family of  $\tau$ -open sets. Suppose that, for*

every  $u \in \Omega$ ,  $\dim A_u \leq n$ . Then there exists a definable  $\tau$ -open set  $B \subseteq \cup_{u \in \Omega} A_u$  such that  $\dim B \leq n$  and  $\cup_{u \in \Omega} A_u \subseteq cl_\tau B$ .

*Proof.* We proceed by induction on  $m$ .

Suppose that  $m = 1$ . By letting  $A_u = \emptyset$  for every  $u \in M \setminus \Omega$  if necessary, we may assume that  $\Omega = M$ . The result then follows from Lemma 3.2.3.

Now suppose that  $m > 1$ . Let  $\mathcal{A} = \{A_u : u \in \Omega\}$ . For every  $x \in \pi(\Omega)$  let  $\Omega_x = \{t \in M : \langle x, t \rangle \in \Omega\}$ , and consider the family  $\mathcal{A}_x = \{A_{x,t} : t \in \Omega_x\}$ . By the case  $m = 1$  there exists a definable  $\tau$ -open set  $B_x \subseteq \cup \mathcal{A}_x$  such that  $\dim B_x \leq n$  and  $\cup \mathcal{A}_x \subseteq cl_\tau B_x$ .

By Lemma 3.2.3, note that the sets  $B_x$  may be chosen definable over the same parameters as  $\mathcal{A}_x$  and  $\tau$ . Hence, by applying the usual argument involving model theoretic compactness, we may choose the sets  $B_x$  definably in  $x \in \pi(\Omega)$ .

Finally, by induction hypothesis applied to the family  $\{B_x : x \in \pi(\Omega)\}$  we derive that there exists a definable  $\tau$ -open set  $B$  with  $\dim B \leq n$  and  $\cup_{x \in \pi(\Omega)} B_x \subseteq cl_\tau B$ . In particular

$$\bigcup_{u \in \Omega} A_u \subseteq \bigcup_{x \in \pi(\Omega)} cl_\tau B_x \subseteq cl_\tau \left( \bigcup_{x \in \pi(\Omega)} B_x \right) \subseteq cl_\tau B. \quad \square$$

Recall the notion of local dimension  $\dim_x(X, \tau)$  of a definable topological space  $(X, \tau)$  at a point  $x$  (Definition 2.2.19), i.e. if  $\mathcal{B}$  denotes a basis of  $\tau$ -neighborhoods of  $x$  then

$$\dim_x(X, \tau) = \min\{\dim A : A \in \mathcal{B}\}.$$

We introduce the following terminology, which we will use from now on in this section. Let  $(X, \tau)$  be a definable topological space. For any  $m$  let

$$X(m) = \{x \in X : \dim_x(X, \tau) \leq m\}.$$

Note that, for any  $m$ , the set  $X(m)$  is definable and satisfies that  $X(m) \subseteq X(m+1)$ . Moreover if  $\dim X = n$  then, for any  $m \geq n$ , it holds that  $X(m) = X(n) = X$ .

**Remark 3.2.5.** For every  $m$ , the set  $X(m)$  is  $\tau$ -open. To see this let  $x \in X(m)$  and let  $A$  be any definable  $\tau$ -open neighborhood of  $x$  such that  $\dim A \leq m$ . Then clearly  $A \subseteq X(m)$ .

**Theorem 3.2.6.** *Suppose that  $\mathcal{M}$  expands  $(\mathbb{R}, <)$ . Let  $(X, \tau)$  be a definable topological space. The following are equivalent.*

- (1)  $(X, \tau)$  is definably separable.
- (2)  $(X, \tau)$  is separable.
- (3)  $(X, \tau)$  has the countable chain condition.

*Proof.* The implication (2) $\Rightarrow$ (3) is a simple known fact in general topology. The implication (3) $\Rightarrow$ (1) follows directly from Lemma 3.2.2. We prove (1) $\Rightarrow$ (2).

Suppose that  $(X, \tau)$  is definably separable. Let  $n = \dim X$ . We may assume that  $n > 0$ . By Remark 3.2.5 the subspace  $X(n-1)$  is  $\tau$ -open, and so it is clearly definably separable. Let  $\mathcal{B}$  be a definable basis for  $\tau$  and let  $\mathcal{B}' = \{A \in \mathcal{B} : \dim A < n\}$ . Clearly  $X(n-1) = \cup \mathcal{B}'$ . By Lemma 3.2.4 there exists a definable  $\tau$ -open set  $B \subseteq X(n-1)$  such that  $\dim B < n$  and  $X(n-1) \subseteq cl_\tau B$ .

Since  $B$  is  $\tau$ -open it is also definably separable. By induction hypothesis we derive that it is separable. Let  $D_1$  denote a countable dense subset of  $(B, \tau)$ . Since  $X(n-1) \subseteq cl_\tau B$  note that  $D_1$  is also dense in  $X(n-1)$ .

Now recall (Remark 3.1.2) that the space  $X$  is separable in the euclidean topology. Let  $D_2$  denote an  $e$ -dense countable subset of  $X$ .

Finally let  $D = D_1 \cup D_2$ . This is a countable set. Let  $A$  be a definable open set in  $(X, \tau)$ . If  $A \cap X(n-1) \neq \emptyset$  then  $A \cap D_1 \neq \emptyset$ . If  $A \cap (X \setminus X(n-1)) \neq \emptyset$  then, by definition of  $X(n-1)$ , it must be that  $\dim A = n$ . But then, by the frontier dimension inequality of the euclidean topology, we have that  $A$  has interior in the euclidean topology on  $X$ , and consequently  $A \cap D_2 \neq \emptyset$ .  $\square$

From Lemma 3.2.4 we may also derive the following characterization of definable separability.

**Proposition 3.2.7.** *A definable topological space  $(X, \tau)$  is definably separable if and only if there exists a  $\tau$ -open definable dense subset  $Y \subseteq X$  satisfying that, for every  $m$ ,  $\dim Y(m) \leq m$ .*

*In particular if  $(X, \tau)$  has the frontier dimension inequality then it suffices to take  $Y = X$ .*

*Proof.* To derive the second sentence of the proposition from the first one it suffices to show that, if  $(X, \tau)$  has the frontier dimension inequality and has a definable subset  $Y$  as described, then  $\dim X(m) \leq m$  for every  $m$ . This follows from the following observation. For every  $m$  and  $x \in X$ , if  $x \in cl_\tau(Y \setminus Y(m))$  then, by definition of  $Y(m)$ , it holds that  $\dim_x(X, \tau) > m$ . It follows that, for every  $m$ ,  $X(m) \subseteq cl_\tau Y(m)$ . By the frontier dimension inequality and properties of  $Y$ , we have that  $\dim X(m) \leq \dim cl_\tau Y(m) = \dim Y(m) \leq m$ .

We now prove the first sentence in the proposition. We start by proving the “if” direction. Let  $Y$  be as in the proposition and, towards a contradiction, suppose that  $(X, \tau)$  is not definably separable. Let  $\mathcal{A}$  be an infinite family of pairwise disjoint  $\tau$ -open sets. Since  $Y$  is dense note that every  $A \in \mathcal{A}$  satisfies that  $A \cap Y \neq \emptyset$ . Let  $m$  denote the smallest value such that  $A \cap Y(m) \neq \emptyset$  for infinitely many sets  $A \in \mathcal{A}$ . If  $m = 0$  then we derive a contradiction from the facts that  $\dim Y(0) \leq 0$  and the sets in  $\mathcal{A}$  are pairwise disjoint. From now on suppose that  $m > 0$ .

Let  $\mathcal{A}' = \{A \in \mathcal{A} : A \cap (Y(m) \setminus Y(m-1)) \neq \emptyset\}$ . Note that  $\mathcal{A}'$  is infinite and moreover, by definition of  $Y(m) \setminus Y(m-1)$ , every  $A \in \mathcal{A}'$  satisfies that  $\dim A = m$ . A contradiction follows from the fact that  $\dim Y(m) \leq m$  and the Fiber Lemma for o-minimal dimension (Lemma 2.1.3).

We now prove the “only if” implication. Suppose that  $(X, \tau)$  is definably separable. We proceed by induction on  $n = \dim X$ . The case  $n = 0$  is trivial, and so we may assume that  $n > 0$ .

Let  $\mathcal{B}$  be a definable basis for  $\tau$  and let  $\mathcal{B}' = \{A \in \mathcal{B} : \dim A < n\}$ . Clearly  $X(n-1) = \cup \mathcal{B}'$ . By Lemma 3.2.4 there exists a definable  $\tau$ -open set  $Z \subseteq X(n-1)$  with  $\dim Z < n$  such that  $X(n-1) \subseteq cl_\tau Z$ . Since  $\dim Z < n$ , by induction hypothesis there exists some definable subset  $Y' \subseteq Z$  that is open and dense in  $(Z, \tau)$ , and moreover satisfies that  $\dim Y'(m) \leq m$  for every  $m$ . Since  $Z$  is  $\tau$ -open note that  $Y'$  is  $\tau$ -open too. Moreover, since  $Y'$  is dense in  $(Z, \tau)$  and  $Z$  is dense in  $(X(n-1), \tau)$ , we derive that  $Y'$  is dense in  $(X(n-1), \tau)$ .

Now let

$$Y = Y' \cup int_\tau(X \setminus X(n-1)).$$

Clearly  $Y$  is  $\tau$ -open and, since  $Y'$  is dense in  $(X(n-1), \tau)$ ,  $Y$  is dense in  $(X, \tau)$ . Recall that  $Y' \subseteq X(n-1)$ . Moreover, by definition of  $X(n-1)$ , for every  $x \in \text{int}_\tau(X \setminus X(n-1))$  and small enough  $\tau$ -neighborhood  $A$  of  $x$ , it holds that  $\dim A = n$  and  $A \subseteq \text{int}_\tau(X \setminus X(n-1)) \subseteq Y$ . So  $\dim_x(Y, \tau) = n$ . It follows that, for every  $m < n$ ,  $Y(m) = Y'(m)$ . We conclude that  $\dim Y(m) \leq m$  for every  $m$ .  $\square$

We end this chapter by deriving from Proposition 3.2.7 that, much like in general topology, definable separability is maintained after taking products.

**Corollary 3.2.8.** *Let  $(X, \tau)$  and  $(Z, \mu)$  be definably separable definable topological spaces. The product space  $(X \times Z, \tau \times \mu)$  is definably separable.*

*Proof.* Let  $Y_1 \subseteq X$  be as described in Proposition 3.2.7, i.e.  $Y_1$  is a definable  $\tau$ -open set that is dense in  $(X, \tau)$  and satisfies that, for every  $m$ ,  $\dim Y_1(m) \leq m$ . Let  $Y_2 \subseteq Z$  be analogous with respect to  $(Z, \mu)$ . Let  $Y = Y_1 \times Y_2$ .

Clearly  $Y$  is open and dense in  $(X \times Z, \tau \times \mu)$ . We show that, for every  $m$ , it holds that  $\dim Y(m) \leq m$ . By Proposition 3.2.7 we conclude that  $(X \times Z, \tau \times \mu)$  is definably separable.

For any  $x \in Y_1$  and  $z \in Y_2$ , note that  $\dim_{\langle x, z \rangle}(Y, \tau \times \mu) = \dim_x(Y_1, \tau) + \dim_z(Y_2, \mu)$ . It follows that, for any  $m$ , the set  $Y(m)$  is the union of sets of the form  $Y(m') \times Y(m'')$  for  $m' + m'' = m$ . Since  $\dim(Y(m') \times Y(m'')) = \dim Y(m') + \dim Y(m'') \leq m' + m''$ , we conclude that  $\dim Y(m) \leq m$ .  $\square$

## 4. DEFINABLE FIRST COUNTABILITY

### Introduction

In this chapter we study directed sets definable in o-minimal expansions of groups and fields. We make the assumption throughout that  $\mathcal{M}$  expands an ordered group. By “directed set” we mean a preordered set in which every finite subset has a lower (if downward directed) or upper (if upward directed) bound.

We show that, in expansions of ordered fields, definable directed sets admit certain co-final definable curves, as well as a suitable analogue in expansions of ordered groups, and furthermore that no analogue holds in full generality. We use the theory of tame pairs to extend the results in the field case to definable families of sets with the finite intersection property. We then apply our results to the study of definable topologies. We prove that, in o-minimal expansions of groups and fields, all definable topological spaces display definable properties akin to first countability.

In Section 4.1 we introduce the necessary definitions. In Section 4.2 we prove the main result on definable directed sets (Theorem 4.2.2) and show that it does not hold in all o-minimal structures. In Section 4.3 we strengthen the main result in the case where the underlying structure expands an ordered field (Corollary 4.3.4). In Section 4.4 we apply the theory of tame pairs to make some additional remarks and frame our work in the context of types (Theorem 4.4.2). We also strengthen our earlier results in the case where the underlying structure expands an archimedean field (Corollary 4.4.9). In Section 4.5 we use the results in previous sections to describe definable bases of neighborhoods of points in a definable topological space (definable first countability, Theorem 4.5.2) and derive some consequences of this, in particular showing that, whenever the underlying o-minimal structure expands an ordered field, definable topological spaces admit definable curve selection (Lemma 4.5.4).

For papers treating objects similar to definable directed sets, namely orders and partial orders, the reader may consult [42] and [43], in which the authors prove, respectively, that definable orders in o-minimal expansions of groups are lexicographic orders (up to definable order-isomorphism), and that definable partial orders in o-minimal structures are extendable to definable total orders. We do not however use these results in this chapter.



This chapter is based on joint work by the author [2].

## 4.1 Directed sets and families

In this chapter  $\mathcal{M}$  is an o-minimal expansion of an ordered group.

Recall that a *preorder* is a transitive and reflexive binary relation.

**Definition 4.1.1.** *A definable preordered set is a definable set  $\Omega \subseteq M^n$  together with a definable preorder  $\preceq$  on  $\Omega$ . A definable downward directed set is a definable preordered set  $(\Omega, \preceq)$  that is downward directed, i.e. for every finite subset  $\Omega' \subseteq \Omega$  there exists  $v \in \Omega$  such that  $v \preceq u$  for all  $u \in \Omega'$ .*

Recall that a subset  $S$  of a preordered set  $(\Omega, \preceq)$  is cofinal if, for every  $u \in \Omega$ , there exists  $v \in S$  such that  $u \preceq v$ . We refer to this property as *upward cofinality*, and work mostly with the dual notion of coinitiality, which we in turn refer to as *downward cofinality*. The reason for this approach is that it seems more natural for the later application of our results to definable topologies.

**Definition 4.1.2.** *Let  $(\Omega', \preceq')$  and  $(\Omega, \preceq)$  be preordered sets. Given  $S \subseteq \Omega$ , a map  $\gamma : \Omega' \rightarrow \Omega$  is downward cofinal for  $S$  (with respect to  $\preceq'$  and  $\preceq$ ) if, for every  $u \in S$ , there exists  $v = v(u) \in \Omega'$  such that  $w \preceq' v$  implies  $\gamma(w) \preceq u$ . Equivalently, we say that  $\gamma : (\Omega', \preceq') \rightarrow (\Omega, \preceq)$  is downward cofinal for  $S$ . We say that  $\gamma$  is downward cofinal if it is downward cofinal for  $\Omega$ .*

*We say that a curve  $\gamma : (0, \infty) \rightarrow \Omega$  is downward cofinal for  $S \subseteq \Omega$  if  $\gamma : ((0, \infty), \leq) \rightarrow (\Omega, \preceq)$  is. If there exists such a map when  $S = \Omega$ , then we may say that  $(\Omega, \preceq)$  admits a downward cofinal curve.*

The dual notion of definable downward directed set is that of *definable upward directed set*. In other words, if  $(\Omega, \preceq)$  is a preordered set and  $\preceq^*$  is the dual preorder of  $\preceq$ , then  $(\Omega, \preceq)$  is a definable upward directed set whenever  $(\Omega, \preceq^*)$  is a definable downward directed set. Moreover, given a preordered set  $(\Omega', \preceq')$ , a map  $\gamma : (\Omega', \preceq') \rightarrow (\Omega, \preceq)$  is *upward cofinal* for  $S \subseteq \Omega$  if it is downward cofinal for  $S$  with respect to  $\preceq'$  and  $\preceq^*$ .

Note that the image of a downward (respectively upward) cofinal map is always downward (respectively upward) cofinal.

Recall that curves are classically defined to be any map into  $M^n$  with interval domain (Definition 2.2.11, taken from [17]), however since in this section we are assuming throughout that  $\mathcal{M}$  expands an ordered group, we restrict our notion of curve to those with domain equal to  $(0, \infty)$ , since, in the o-minimal group setting, these are equivalent for all practical purposes (e.g. Proposition 2.2.14) to the general class of curves with interval domain. Additionally our notion of downward/upward cofinal curve focuses on the behaviour of curves as  $t \rightarrow 0^+$ . Cofinal curves however will only be relevant in this chapter in the setting where the underlying structure expands an ordered field, where once again this notion of convergence is strong enough for all purposes.

**Remark 4.1.3.**

- (i) Let  $\mathcal{S} = \{S_u : u \in \Omega\}$  be a definable family of sets. Set inclusion induces a definable preorder  $\preceq_{\mathcal{S}}$  on  $\Omega$  given by  $u \preceq_{\mathcal{S}} v \Leftrightarrow S_u \subseteq S_v$ .
- (ii) Conversely, let  $(\Omega, \preceq)$  be a definable preordered set. Consider the definable family of nonempty sets  $\{S_u : u \in \Omega\}$ , where  $S_u = \{v \in \Omega : v \preceq u\}$  for every  $u \in \Omega$ . Then, for every  $u, w \in \Omega$ ,  $u \preceq w$  if and only if  $S_u \subseteq S_w$ .

Note that Remark 4.1.3 remains true if we drop the word “definable” from the statements. Motivated by Remark 4.1.3(i) we introduce the following definition.

**Remark 4.1.4.** Note that a family of sets  $\mathcal{S} = \{S_u : u \in \Omega\}$  is *downward* (respectively *upward*) *directed* if  $\emptyset \notin \mathcal{S}$  and the preorder  $\preceq_{\mathcal{S}}$  on  $\Omega$  induced by set inclusion in  $\mathcal{S}$  forms a downward (respectively upward) directed set.

In other words,  $\mathcal{S}$  is downward (respectively upward) directed if and only if  $\emptyset \notin \mathcal{S}$  and, for every finite  $\mathcal{F} \subseteq \mathcal{S}$ , there exists  $u = u(\mathcal{F}) \in \Omega$  satisfying  $S_u \subseteq S$  (respectively  $S \subseteq S_u$ ) for every  $S \in \mathcal{F}$ .

**Example 4.1.5.** Let  $(\Omega, \preceq)$  be a downward (respectively upward) directed set. The induced family  $\mathcal{S} = \{S_u : u \in \Omega\}$  described in Remark 4.1.3(ii) is a downward (respectively upward) directed family.

**Definition 4.1.6.** Let  $(\Omega, \preceq)$  be a definable preordered set and let  $f : \Omega \rightarrow M^m$  be a definable injective map. We define the push-forward of  $(\Omega, \preceq)$  by  $f$  to be the definable preordered set  $(f(\Omega), \preceq_f)$  that satisfies:  $f(x) \preceq_f f(y)$  if and only if  $x \preceq y$ , for all  $x, y \in \Omega$ . Thus  $(f(\Omega), \preceq_f)$  is the unique definable preordered set such that  $f : (\Omega, \preceq) \rightarrow (f(\Omega), \preceq_f)$  is a preorder-isomorphism.

Let  $\{S_u : u \in \Omega\}$  be a definable family of sets. Based on the correspondence between definable families of sets and definable preordered sets given by Remark 4.1.3 we also define the push-forward of  $\{S_u : u \in \Omega\}$  by  $f$  to be the reindexing of said family given by  $\{S_{f^{-1}(u)} : u \in f(\Omega)\}$ . We abuse notation and write  $S_u$  instead of  $S_{f^{-1}(u)}$  when it is clear that  $u \in f(\Omega)$ .

## 4.2 Main result on directed sets

Throughout recall that  $\|\cdot\| : M^n \rightarrow M$  denotes the usual  $l_\infty$  norm, i.e.

$$\|(x_1, \dots, x_n)\| = \max\{|x_1|, \dots, |x_n|\}.$$

For any  $x \in M^n$  and  $\varepsilon > 0$ , set

$$B(x, \varepsilon) := \{y \in M^n : \|x - y\| < \varepsilon\}$$

and

$$\overline{B(x, \varepsilon)} := \{y \in M^n : \|x - y\| \leq \varepsilon\}$$

to be respectively the open and closed ball of center  $x$  and radius  $\varepsilon$ .

**Definition 4.2.1.** We equip  $M^{>0} \times M^{>0}$  with the definable preorder given by

$$(s, t) \trianglelefteq (s', t') \Leftrightarrow s \leq s' \text{ and } t \geq t'.$$

Note that  $(M^{>0} \times M^{>0}, \trianglelefteq)$  is a definable downward directed set.

We are now ready to state the main theorem of this section.

**Theorem 4.2.2.** *Let  $(\Omega, \preceq)$  be a definable downward (respectively upward) directed set. There exists a definable downward (respectively upward) cofinal map  $\gamma : (M^{>0} \times M^{>0}, \preceq) \rightarrow (\Omega, \preceq)$ .*

Note that, if  $(\Omega, \preceq)$  is definable over a set of parameters  $A$ , and  $b \in M$  is any non-zero parameter, then the downward (respectively upward) cofinal map given by Theorem 4.2.2 is  $Ab$ -definable. To see this, it suffices to recall that the structure  $(\mathcal{M}, b)$  has definable choice, where every choice function is 0-definable. Equivalently any choice function in  $\mathcal{M}$  is  $b$ -definable. Then apply the theorem to the prime model over  $Ab$ , which by definable choice is  $dcl(Ab)$ .

We prove the downward case of Theorem 4.2.2. From the duality of the definitions it is clear that the upward case follows from this. Therefore, since there will be no room for confusion, from now on and until the end of the section we write “directed” and “cofinal” instead of respectively “downward directed” and “downward cofinal”.

In order to prove Theorem 4.2.2 we require three lemmas, the first of which concerns definable families with the finite intersection property. Recall that a family of sets  $\mathcal{S}$  has the finite intersection property (FIP) if the intersection  $\bigcap_{1 \leq i \leq n} S_i$  is nonempty for all  $S_1, \dots, S_n \in \mathcal{S}$ .

**Example 4.2.3.** If  $\mathcal{S}$  is a directed family of sets (e.g. Example 4.1.5), then  $\mathcal{S}$  has the finite intersection property. The converse is not necessarily true; the definable family  $\{M \setminus \{x\} : x \in M\}$  has the finite intersection property but is not directed.

**Lemma 4.2.4.** *Let  $\mathcal{S} = \{S_u \subseteq M^n : u \in \Omega\}$  be a definable family of sets with the finite intersection property. There exists a definable set  $\Omega_h$  and a definable bijection  $h : \Omega \rightarrow \Omega_h$  such that the push-forward of  $\mathcal{S}$  by  $h$  satisfies the following properties.*

- (1) *For every  $u \in \Omega_h$ , there exists  $\varepsilon = \varepsilon(u) > 0$  such that  $\bigcap_{v \in \Omega_h, \|v-u\| < \varepsilon} S_v \neq \emptyset$ .*
- (2) *For every closed and bounded definable set  $B \subseteq \Omega_h$ , there exists  $\varepsilon = \varepsilon(B) > 0$  such that  $\bigcap_{v \in B, \|v-u\| < \varepsilon} S_v \neq \emptyset$  for every  $u \in B$ .*

*Proof.* We prove the result by showing that, for  $\Omega$  and  $\mathcal{S}$  as in the statement of the lemma, there exists a definable map  $f : \Omega \rightarrow \cup_{u \in \Omega} S_u \times M^{>0}$ , given by  $u \mapsto (x_u, \varepsilon_u)$ , such that, for all  $u, v \in \Omega$ ,

$$\|f(u) - f(v)\| < \frac{\varepsilon_u}{2} \Rightarrow x_u \in S_v. \quad ((\dagger)_{(f, \mathcal{S})})$$

If  $(\dagger)_{(f, \mathcal{S})}$  holds for a continuous function  $f$ , then the lemma holds with  $\Omega_h = \Omega$  and  $h = id$ . This follows from the observation that, by  $(\dagger)_{(f, \mathcal{S})}$  and the continuity of  $f$  at  $u \in \Omega$ , we have

$$x_u \in \bigcap_{\substack{v \in \Omega \\ \|u-v\| < \delta}} S_v \neq \emptyset$$

whenever  $\delta > 0$  is sufficiently small. Moreover, if  $f$  is continuous on a definable closed and bounded set  $B \subseteq \Omega$  and  $\varepsilon_{min}$  is the minimum of the map  $u \mapsto \varepsilon_u$  on  $B$ , then uniform continuity yields a  $\delta > 0$  such that  $\|f(u) - f(v)\| < \frac{\varepsilon_{min}}{2}$  for all  $u, v \in B$  satisfying  $\|u - v\| < \delta$ . So in this case we have

$$x_u \in \bigcap_{\substack{v \in B \\ \|u-v\| < \delta}} S_v \neq \emptyset, \quad \text{for all } u \in B.$$

In the case that  $(\dagger)_{(f, \mathcal{S})}$  holds for a function  $f$  which is not necessarily continuous, then we may modify the above argument by identifying an appropriate bijection  $h$  and push-forward  $\mathcal{S}_h = \{S_{h^{-1}(u)} : u \in \Omega_h\}$  to complete the proof as follows. By o-minimality, let  $C_1, \dots, C_l$  be a cell partition of  $\Omega$  such that  $f$  is continuous on every  $C_i$ . Consider the disjoint union  $\Omega_h = \cup_{1 \leq i \leq l} (C_i \times \{i\})$  and the natural bijection  $h : \Omega \rightarrow \Omega_h$ . This map is clearly definable. Moreover, for every cell  $C_i$ , the restriction  $h|_{C_i}$  is a homeomorphism and  $h(C_i)$  is open in  $\Omega_h$ . It follows that the map  $f \circ h^{-1} : \Omega_h \rightarrow \cup_{u \in \Omega} S_u \times M^{>0}$  is definable and continuous. Additionally, note that  $(\dagger)_{(f \circ h^{-1}, \mathcal{S}_h)}$  holds. Consequently, the lemma holds via an analogous argument to the one above.

Thus it remains to prove the existence of a definable function  $f$  satisfying  $(\dagger)_{(f,S)}$ . First of all we show that we can assume that there exists  $0 \leq m \leq n$  such that the following property (P) holds:

$$\dim \left( \bigcap_{1 \leq i \leq k} S_{u_i} \right) = m, \text{ for every } u_1, \dots, u_k \in \Omega. \quad (\text{P})$$

Let  $m'$  be the minimum natural number such that there exist  $u_1, \dots, u_k \in \Omega$  with  $\dim(\bigcap_{1 \leq i \leq k} S_{u_i}) = m'$ . Set  $S := \bigcap_{1 \leq i \leq s} S_{u_i}$ . Consider the definable family  $\mathcal{S}_* := \{S_u \cap S : u \in \Omega\}$ . This family has the FIP. Note that if  $(\dagger)_{(f,\mathcal{S}_*)}$  is satisfied for some definable function  $f$  then  $(\dagger)_{(f,S)}$  is satisfied too. Hence, by passing to  $\mathcal{S}_*$  if necessary, we may assume that (P) holds for some fixed  $m$ .

Suppose that  $m = n$ , so in particular all sets  $S_u$  have nonempty interior (we call this the open case). By definable choice, let  $f : \Omega \rightarrow \cup_u S_u \times M^{>0}$ ,  $f(u) = (x_u, \varepsilon_u)$ , be a definable map such that, for every  $u \in \Omega$ , the open ball of center  $x_u$  and radius  $\varepsilon_u$  is contained in  $S_u$ . Then, for any  $u, v \in \Omega$ ,  $\|f(u) - f(v)\| < \frac{\varepsilon_u}{2}$  implies both  $\|x_u - x_v\| < \frac{\varepsilon_u}{2}$  and  $\varepsilon_v > \frac{\varepsilon_u}{2}$ . Hence  $\|x_u - x_v\| < \varepsilon_v$ , and so  $x_u \in S_v$ .

Now suppose that  $m < n$ . Let  $u_0$  be a fixed element in  $\Omega$  and let  $\mathcal{X}$  be a finite partition of  $S_{u_0}$  into cells. We claim that there must exist some cell  $C \in \mathcal{X}$  such that  $\dim(C \cap \bigcap_{1 \leq i \leq k} S_{u_i}) = m$  for any  $u_1, \dots, u_k \in \Omega$ . Suppose that the claim is false. Then, for every  $C \in \mathcal{X}$ , there exist  $k_C < \omega$  and  $u_1^C, \dots, u_{k_C}^C \in \Omega$  such that  $\dim(C \cap \bigcap_{1 \leq i \leq k_C} S_{u_i^C}) < m$ . In that case however

$$\begin{aligned} m &= \dim \left( S_{u_0} \cap \bigcap_{C \in \mathcal{X}, 1 \leq i \leq k_C} S_{u_i^C} \right) \leq \dim \left( \bigcup_{C \in \mathcal{X}} \left( C \cap \bigcap_{1 \leq i \leq k_C} S_{u_i^C} \right) \right) \\ &= \max_{C \in \mathcal{X}} \left\{ \dim \left( C \cap \bigcap_{1 \leq i \leq k_C} S_{u_i^C} \right) \right\} < m, \end{aligned}$$

which is a contradiction. So the claim holds.

Let  $C_0 \in \mathcal{X}$  be a cell with the described property. Clearly  $\dim(C_0) = m$ . Consider the definable family  $\mathcal{S}' = \{S'_u = C_0 \cap S_u : u \in \Omega\}$ . By the claim,  $\mathcal{S}'$  satisfies the FIP; in fact,

any intersection of finitely many sets in  $\mathcal{S}'$  has dimension  $m$ . We prove the lemma for  $\mathcal{S}'$  and the result for  $\mathcal{S}$  follows.

Let  $\pi : C_0 \rightarrow \pi(C_0) \subseteq M^m$  be a projection which homeomorphically maps  $C_0$  onto an open cell  $\pi(C_0)$ . Since  $\pi$  is a bijection, the set  $\pi(S'_u) \subseteq M^m$  has nonempty interior, for every  $u \in \Omega$ . Consider the definable family  $\{\pi(S'_u) \subseteq M^m : u \in \Omega\}$ . This is a definable family of sets with nonempty interior that has the FIP. By the open case, there exists a definable  $g = (g_1, g_2) : \Omega \rightarrow M^m \times M^{>0}$ ,  $g(u) = (x_u, \varepsilon_u)$ , such that, for every  $u, v \in \Omega$ ,  $\|g(u) - g(v)\| < \frac{\varepsilon_u}{2}$  implies  $x_u \in \pi(S'_v)$ , i.e.  $\pi^{-1}(x_u) \in S'_v$ . Let  $f = (f_1, f_2) : \Omega \rightarrow M^m \times M^{>0}$  be given by  $f_1 = \pi^{-1} \circ g_1$  and  $f_2 = g_2$ . Since  $g$  is a projection of  $f$ , we have  $\|g(u) - g(v)\| \leq \|f(u) - f(v)\|$ , for all  $u, v \in \Omega$ . The result follows.  $\square$

**Definition 4.2.5.** Let  $(\Omega, \preceq)$  be a directed set. We say that  $\Omega' \subseteq \Omega$  is  $\preceq$ -bounded in  $\Omega$  if there exists  $v \in \Omega$  such that  $v \preceq u$  for all  $u \in \Omega'$ . We write  $v \preceq \Omega'$ .

Let  $(\Omega, \preceq)$  be a definable directed set and let  $\mathcal{S} = \{S_u : u \in \Omega\}$  be a directed family of sets as in Example 4.1.5. By construction of  $\mathcal{S}$ , note that, if a subfamily  $\{S_u : u \in \Omega'\}$  has nonempty intersection, then  $\Omega'$  is  $\preceq$ -bounded in  $\Omega$ . Hence Lemma 4.2.4 yields the following corollary.

**Corollary 4.2.6.** Any definable directed set  $(\Omega', \preceq')$  is definably preorder-isomorphic to a definable directed set  $(\Omega, \preceq)$  such that

- (1) for all  $u \in \Omega$  there exists an  $\varepsilon > 0$  such that  $B(u, \varepsilon) \cap \Omega$  is  $\preceq$ -bounded in  $\Omega$ ;
- (2) for any definable closed and bounded set  $B \subseteq \Omega$ , there exists an  $\varepsilon > 0$  such that  $B(u, \varepsilon) \cap B$  is  $\preceq$ -bounded in  $\Omega$  for every  $u \in B$ .

We now prove a lemma (Lemma 4.2.8) which allows us to see  $\preceq$ -boundedness as a local property, and which we will use in proving Theorem 4.2.2. We first show that the doubling property of the supremum metric in  $\mathbb{R}^n$  generalizes to  $M^n$ .

**Lemma 4.2.7.** *For any  $x \in M^n$  and  $r > 0$ , there exists a finite set of points  $P \subseteq B(x, r)$ , where  $|P| \leq 3^n$ , such that the sets of balls of radius  $\frac{r}{2}$  centered on points in  $P$  covers the ball of radius  $r$  centered on  $x$ , i.e.*

$$B(x, r) \subseteq \bigcup_{y \in P} B\left(y, \frac{r}{2}\right).$$

*Proof.* Fix  $x = (x_1, \dots, x_n) \in M^n$  and  $r > 0$ . Set

$$P := \left\{ y = (y_1, \dots, y_n) : y_i = x_i + \delta_i, \delta_i \in \left\{ \frac{r}{2}, -\frac{r}{2}, 0 \right\}, 1 \leq i \leq n \right\}.$$

Note that  $|P| = 3^n$  and, for every  $z \in B(x, r)$ , there exists some  $y \in P$  such that  $\|y - z\| < \frac{r}{2}$ . Thus  $B(x, r) \subseteq \bigcup_{y \in P} B(y, \frac{r}{2})$ .  $\square$

**Lemma 4.2.8.** *Let  $(\Omega, \preceq)$  be a definable directed set and let  $S \subseteq \Omega$  be a bounded definable set. Suppose that there exists  $\varepsilon_0 > 0$  such that, for all  $u \in S$ ,  $B(u, \varepsilon_0) \cap S$  is  $\preceq$ -bounded in  $\Omega$ . Then  $S$  is  $\preceq$ -bounded in  $\Omega$ .*

*Proof.* Consider the definable set

$$H = \{ \varepsilon : \forall u \in S, B(u, \varepsilon) \cap S \text{ is } \preceq\text{-bounded in } \Omega \}.$$

We have  $(0, \varepsilon_0) \subseteq H$ , and so  $H$  is nonempty, and hence by o-minimality it must be of the form  $(0, r)$ , for some  $r \in [\varepsilon_0, \infty) \cup \{\infty\}$ . We show  $H = (0, \infty)$  and thus, since  $S$  is bounded, that  $S$  is  $\preceq$ -bounded in  $\Omega$ .

Suppose that  $r < \infty$ . We reach a contradiction by showing  $\frac{4}{3}r \in H$ . To do so we fix an arbitrary  $u_0 \in S$  and show that  $B(u_0, \frac{4}{3}r) \cap S$  is  $\preceq$ -bounded in  $\Omega$ .

By repeated application of Lemma 4.2.7, the ball  $B(u_0, \frac{4}{3}r)$  can be covered by finitely many ( $\leq 3^{2n}$ ) balls  $B_1, \dots, B_k$  of radius  $\frac{r}{3}$ . For any  $1 \leq i \leq k$ , if  $w \in B_i \cap S$ , then  $B_i \subseteq B(w, \frac{2}{3}r)$ , by the triangle inequality. By assumption,  $B(w, \frac{2}{3}r) \cap S$  is  $\preceq$ -bounded in  $\Omega$ , and so the set  $B_i \cap S$  is  $\preceq$ -bounded in  $\Omega$ . Moreover, if  $B_i \cap S$  is empty for some  $1 \leq i \leq k$ , then  $B_i \cap S$  is trivially  $\preceq$ -bounded in  $\Omega$ . By the definition of directed set, it



follows that  $\cup_i B_i \cap S$  is  $\preceq$ -bounded in  $\Omega$ , and so, since  $B(u_0, \frac{4}{3}r) \subseteq \cup_i B_i$ , the set  $B(u_0, \frac{4}{3}r) \cap S$  is  $\preceq$ -bounded in  $\Omega$ .  $\square$

Corollary 4.2.6 and Lemma 4.2.8 together yield the following.

**Corollary 4.2.9.** *Let  $(\Omega', \preceq')$  be a definable directed set. Then there exists a definable directed set  $(\Omega, \preceq)$  that is definably preorder isomorphic to  $(\Omega', \preceq')$  and such that every closed and bounded definable subset of  $\Omega$  is  $\preceq$ -bounded in  $\Omega$ .*

We now borrow a definition from [13].

**Definition 4.2.10.** *A definable set  $S$  is  $D_\Sigma$  if there exists a definable family of closed and bounded sets  $\{S(s, t) : (s, t) \in M^{>0} \times M^{>0}\}$  such that  $S = \cup_{s, t} S(s, t)$  and  $S(s, t) \subseteq S(s', t')$  whenever  $(s', t') \preceq (s, t)$ .*

To prove Theorem 4.2.2 we use the fact that every definable set is  $D_\Sigma$ . This can be derived from [13]. We include a proof for the sake of completeness.

**Lemma 4.2.11.** *Every definable set is  $D_\Sigma$ .*

*Proof.* Let  $S$  denote a definable set. We proceed by induction on  $\dim(S)$ .

If  $S$  is closed, then define  $S(s, t) = \overline{B(0, t)} \cap S$ , for every  $(s, t) \in M^{>0} \times M^{>0}$ . Clearly every  $S(s, t)$  is closed and bounded and  $\cup_{s, t} S(s, t) = S$ . Moreover,  $S(s, t) \subseteq S(s', t')$  whenever  $t \leq t'$ . So  $S$  is  $D_\Sigma$ . In particular, if  $\dim(S) \leq 0$ , then  $S$  is  $D_\Sigma$ .

Now suppose that  $\dim(S) \geq 1$ . By the above, we may assume that  $S$  is not closed. Set  $\overline{\partial S} := cl(\partial S)$ . By o-minimality,  $\dim(\overline{\partial S}) < \dim(S)$  and so  $S \cap \overline{\partial S}$  is  $D_\Sigma$  by induction. Let  $S_0 = S \setminus \overline{\partial S}$ . Since the union of finitely many closed and bounded sets is closed and bounded, one may easily deduce that the union of finitely many  $D_\Sigma$  sets is  $D_\Sigma$ . Hence to prove that  $S$  is  $D_\Sigma$  it suffices to show that  $S_0$  is  $D_\Sigma$ .

For every  $s > 0$ , set

$$\overline{\partial S}(s) := \bigcup \{B(x, s) : x \in \overline{\partial S}\}.$$

Note that, since  $\partial S_0 \subseteq cl(S) \setminus S_0 = \overline{\partial S}$ , for every  $s > 0$  it holds that  $S_0 \setminus \overline{\partial S}(s) = cl(S_0) \setminus \overline{\partial S}(s)$ , meaning that  $S_0 \setminus \overline{\partial S}(s)$  is closed. For any  $(s, t) \in M^{>0} \times M^{>0}$ , let  $S(s, t) =$

$\overline{B(0,t)} \cap S_0 \setminus \overline{\partial S}(s)$ . Every set  $S(s,t)$  is closed and bounded and  $S(s,t) \subseteq S(s',t')$  whenever  $(s',t') \trianglelefteq (s,t)$ . Moreover, since  $\overline{\partial S}$  is closed, for every  $x \in S_0$  there exists  $s > 0$  such that  $B(x,s) \cap \overline{\partial S} = \emptyset$ , so  $x \notin \overline{\partial S}(s)$ , and in particular  $x \in S(s,t)$  for all sufficiently large  $t > 0$ . Hence  $\cup_{s,t} S(s,t) = S$ . So  $S_0$  is  $D_\Sigma$ .  $\square$

We now complete the proof of Theorem 4.2.2.

*Proof of Theorem 4.2.2.* Let  $(\Omega, \preceq)$  be a definable directed set. We construct a definable cofinal map  $\gamma : (M^{>0} \times M^{>0}, \trianglelefteq) \rightarrow (\Omega, \preceq)$ .

Clearly it is enough to prove the statement for any definable preorder definably preorder-isomorphic to  $(\Omega, \preceq)$ . Hence, by Corollary 4.2.9, we may assume that any definable closed and bounded subset of  $\Omega$  is  $\preceq$ -bounded.

Lemma 4.2.11 yields a definable family  $\{\Omega(s,t) : (s,t) \in M^{>0} \times M^{>0}\}$  of closed and bounded sets such that  $\Omega = \cup_{s,t} \Omega(s,t)$  and  $\Omega(s,t) \subseteq \Omega(s',t')$  whenever  $(s',t') \trianglelefteq (s,t)$ . By assumption on  $\Omega$ , every  $\Omega(s,t)$  is  $\preceq$ -bounded. Applying definable choice let  $\gamma : M^{>0} \times M^{>0} \rightarrow \Omega$  be a definable map satisfying  $\gamma(s,t) \preceq \Omega(s,t)$ , for every  $(s,t) \in M^{>0} \times M^{>0}$  (if  $\Omega(s,t)$  is empty then trivially any value  $\gamma(s,t) \in \Omega$  will do). For every  $x \in \Omega$ , there exists  $(s_x, t_x) \in M^{>0} \times M^{>0}$  such that  $x \in \Omega(s_x, t_x) \subseteq \Omega(s,t)$ , for all  $(s,t) \trianglelefteq (s_x, t_x)$ . We conclude that  $\gamma$  is cofinal.  $\square$

Note that Theorem 4.2.2 implies that, if  $(M, <)$  is separable, then every definable directed set has countable cofinality. In Proposition 4.5.1 we show that this holds in the greater generality of any o-minimal structure.

**Remark 4.2.12.** Given a definable directed set  $(\Omega, \preceq)$ , one may ask whether or not the map  $\gamma : (M^{>0} \times M^{>0}, \trianglelefteq) \rightarrow (\Omega, \preceq)$  given by Theorem 4.2.2 may always be chosen such that  $\gamma(M^{>0} \times M^{>0})$  is totally ordered by  $\preceq$ . The answer is no. Consider the definable family  $\{(0,t) \cup (2t, 3t) : t > 0\}$ . Following Remark 4.1.3(i), set inclusion in this family defines a directed set  $(M^{>0}, \preceq)$ , where  $t \preceq t'$  iff  $t = t'$  or  $3t \leq t'$ . It is easy to see that no infinite definable subset of  $M^{>0}$  is totally ordered by  $\preceq$ .

Applying first-order compactness and definable choice in the usual fashion we may derive a uniform version of Theorem 4.2.2. We leave the details to the reader.

**Corollary 4.2.13.** *Let  $\{(\Omega_x, \preceq_x) : \Omega_x \subseteq M^m, x \in \Sigma \subseteq M^n\}$  be a definable family of downward directed sets. There exists a definable family of functions*

$$\{\gamma_x : (M^{>0} \times M^{>0}, \trianglelefteq) \rightarrow (\Omega_x, \preceq_x) : x \in \Sigma\}$$

*such that, for every  $x \in \Sigma$ ,  $\gamma_x$  is downward cofinal.*

We end this section with an example of an o-minimal structure in which Theorem 4.2.2 does not hold. For the rest of the section we drop the assumption that  $\mathcal{M}$  expands a group but keep the assumption that it is an o-minimal expansion of a dense linear order without endpoints. We will consider the property that any definable function from  $M$  to itself is piecewise either constant or the identity (think of  $\mathcal{M}$  as having trivial definable closure, say; for example  $\mathcal{M} = (M, <)$  is a dense linear order without endpoints).

Theorem 4.2.2 tells us that, under the assumption that  $\mathcal{M}$  expands an ordered group, any definable upward directed set has a definable cofinal subset of dimension at most 2 (and analogously for definable downward directed sets).

For every  $n > 0$ , we prove, under the assumption of the above property on  $\mathcal{M}$ , the existence of a definable upward directed set that admits no definable cofinal subset of dimension less than  $n$ . Thus Theorem 4.2.2 does not hold in general for o-minimal structures, even if we substitute  $(M^{>0} \times M^{>0}, \trianglelefteq)$  with any other definable preordered set.

We begin with some notation. For any  $a \in M$  and  $n > 0$ , let

$$X(a, n) = \{(x_1, \dots, x_n) \in M^n : a < x_1 < \dots < x_n\}.$$

Then  $X(a, n)$  is definable and  $n$ -dimensional. We endow  $X(a, n)$  with the definable lexicographic order and show that any definable cofinal subset of  $X(a, n)$  has dimension  $n$ . Hence for the remainder of the section ‘cofinal’ will refer to the lexicographic order.

**Proposition 4.2.14.** *Suppose that any definable map from  $M$  to itself is piecewise constant or the identity. Then any cofinal definable subset of  $X(a, n)$  is  $n$ -dimensional.*

Let  $\pi$  denote the projection onto the first coordinate. We will make use of the following easily derivable fact:

**Fact 4.2.15.** *For any  $m > 0$  and  $b \in M$ , a definable subset  $Y \subseteq X(b, m)$  is cofinal if and only if  $(r, \infty) \subseteq \pi(Y)$  for some  $r > 0$ .*

We now prove Proposition 4.2.14.

*Proof.* We proceed by induction on  $n$ .

Fix  $a \in M$  and  $n > 0$  and let  $X = X(a, n)$ . If  $n = 1$  then the result is trivial. For the inductive step suppose that  $n > 1$ . We assume that there exists a definable cofinal subset  $Y \subset X$  with  $\dim Y < n$  and arrive at a contradiction.

Fact 4.2.15 implies that  $Y_t = \{x \in M^{n-1} : (t, x) \in Y\}$  must be nonempty for sufficiently large  $t > 0$ . As  $\dim Y < n$ , the Fiber Lemma for o-minimal dimension (Lemma 2.1.3) implies the existence of some  $t_0 > 0$  such that  $0 \leq \dim Y_t < n - 1$ , for all  $t > t_0$ . Now note that, for every  $t > a$ , the fiber  $X_t$  is the space  $X(t, n - 1)$ . Applying induction we conclude that  $Y_t$  is not cofinal in  $X_t$  for any  $t > t_0$ . By Fact 4.2.15 and the fact that  $Y_t$  is nonempty, it follows that  $\pi(Y_t)$  has a supremum, which clearly must be greater than  $t$ . Consider the definable map  $f : (t_0, \infty) \rightarrow M$  given by  $t \mapsto \sup \pi(Y_t)$ . This map is well defined and, for every  $t$ ,  $f(t) > t$ . This contradicts the fact that  $f$  is piecewise constant or the identity.  $\square$

### 4.3 A strengthening of the main result for expansions of ordered fields

We recall that a *pole* is a definable bijection  $f : I \rightarrow J$  between a bounded interval  $I$  and an unbounded interval  $J$ . Edmundo proved the following in [20] (see Fact 1.6).

**Fact 4.3.1.** *The structure  $\mathcal{M}$  has a pole if and only if there exist definable functions  $\oplus, \otimes : M^2 \rightarrow M$  such that  $(M, <, \oplus, \otimes)$  is a real closed ordered field.*

The following corollary provides a condition on the structure  $\mathcal{M}$  under which Theorem 4.2.2 can be strengthened to state that every definable directed set admits a definable cofinal curve. From now on, we write that  $\mathcal{M}$  “expands an ordered field” when the conclusion of Fact 4.3.1 holds.

**Corollary 4.3.2.** *The structure  $\mathcal{M}$  has a pole if and only if every definable downward (respectively upward) directed set  $(\Omega, \preceq)$  admits a definable downward (respectively upward) cofinal curve.*

*Proof.* Let  $(\Omega, \preceq)$  be a definable downward (respectively upward) directed set. If  $\gamma_0 : ((0, \infty), \leq) \rightarrow (M^{>0} \times M^{>0}, \trianglelefteq)$  is a downward cofinal curve and  $\gamma_1 : (M^{>0} \times M^{>0}, \trianglelefteq) \rightarrow (\Omega, \preceq)$  is a downward (respectively upward) cofinal map then  $\gamma = \gamma_1 \circ \gamma_0$  is a downward (respectively upward) cofinal curve in  $(\Omega, \preceq)$ . Since such a downward (respectively upward) cofinal map  $\gamma_1$  exists by Theorem 4.2.2, in order to prove the corollary it suffices to show that  $\mathcal{M}$  has a pole if and only if  $(M^{>0} \times M^{>0}, \trianglelefteq)$  admits a definable downward cofinal curve. As usual, from now on throughout the proof we omit the word downward as there is no room for confusion.

Suppose that  $\mathcal{M}$  contains a pole  $f : I \rightarrow J$ , with  $I$  a bounded interval and  $J$  an unbounded interval in  $M$ . By o-minimality, we may assume that  $f$  is strictly monotonic. By transforming  $f$  if necessary by means of the group operation, we may assume that  $I = (0, r)$  for some  $r > 0$  and that  $\lim_{t \rightarrow 0^+} f(t) = \infty$ . Continue  $f$  to  $(0, \infty)$  by setting  $f$  to be zero on  $[r, \infty)$ . For some fixed  $a > 0$ , define  $f_a : (0, \infty) \rightarrow M$  by  $f_a(t) = \max\{f(t), a\}$ . The curve  $\gamma$  given by  $\gamma(t) = (t, f_a(t))$  for all  $t > 0$  is clearly a definable cofinal curve in  $(M^{>0} \times M^{>0}, \trianglelefteq)$ .

Conversely, suppose that  $\gamma$  is a definable cofinal curve in  $(M^{>0} \times M^{>0}, \trianglelefteq)$  and let  $\Gamma = \gamma[(0, \infty)]$ . Let  $\pi : M^2 \rightarrow M$  denote the projection onto the first coordinate. Since  $\gamma$  is cofinal, we have  $(0, s_1) \subseteq \pi(\Gamma)$  for some  $s_1 > 0$ . Since  $\Gamma$  is one-dimensional, by the Fiber Lemma for o-minimal dimension there exists  $s_2 > 0$  such that, for every  $0 < s < s_2$ , the fiber  $\Gamma_s$  is finite. Let  $s_0 = \min\{s_1, s_2\}$  and consider the definable map  $\mu : (0, s_0) \rightarrow M$  given by  $\mu(s) = \max \Gamma_s$ . This map is well defined. Since  $\gamma$  is cofinal for every  $s', t' > 0$ , there exists  $s, t > 0$ , with  $s \leq s'$  and  $t \geq t'$ , such that  $(s, t) \in \Gamma$ . It follows that  $\lim_{s \rightarrow 0^+} \mu(s) = \infty$ . By o-minimality, there exists some interval  $(0, r) \subseteq (0, s_0)$  such that  $\mu|_{(0, r)} : (0, r) \rightarrow (\mu(r), \infty)$  is a pole.  $\square$

**Remark 4.3.3.** The proof above shows that, if there exists a definable one-dimensional pre-ordered set  $(\Sigma, \preceq_\Sigma)$  and a definable downward cofinal map  $\gamma_\Sigma : (\Sigma, \preceq_\Sigma) \rightarrow (M^{>0} \times M^{>0}, \trianglelefteq)$ , then  $\mathcal{M}$  has a pole. Hence Theorem 4.2.2 cannot be “improved” by putting a one-dimensional space in place of  $(M^{>0} \times M^{>0}, \trianglelefteq)$ , unless we assume the existence of a pole in the structure, in which case by the above corollary  $(\Sigma, \preceq_\Sigma)$  can always be taken to be  $((0, \infty), \leq)$ .

By Fact 4.3.1, the following is an immediate consequence of Corollary 4.3.2.

**Corollary 4.3.4.** *Every definable downward (respectively upward) directed set  $(\Omega, \preceq)$  admits a definable downward (respectively upward) cofinal curve if and only if  $\mathcal{M}$  expands an ordered field.*

#### 4.4 Tame Extensions

In this section we use the theory of tame pairs, as well as results from previous sections, to obtain results about definable families of sets with the finite intersection property (FIP). We conclude with a strengthening of Corollary 4.3.4 in the case that  $\mathcal{M}$  expands an archimedean field (Corollary 4.4.9). Along the way we prove some facts about types that will be further explored in Chapter 5 (Section 5.2).

Recall the conventions and definitions on types established in Chapter 2 (Section 2.1.4). In particular we refer to types interchangeably as consistent families of formulas and as families of definable sets with the finite intersection property (FIP). Moreover unless otherwise specified all types we consider are global and complete. Recall the definitions of definable type and uniform (type) basis.

**Definition 4.4.1.** *Let  $\mathcal{S}$  and  $\mathcal{F}$  be families of sets. We say that  $\mathcal{F}$  is finer than  $\mathcal{S}$  if, for every  $S \in \mathcal{S}$ , there is  $F \in \mathcal{F}$  such that  $F \subseteq S$ .*

The following is the main result of this section. For the proof we rely on the completeness of the theory of tame pairs. See Chapter 2 (Section 2.1.4) for definitions and relevant facts on tame extensions and pairs.

**Theorem 4.4.2.** *Suppose that  $\mathcal{M}$  expands an ordered field,  $\Omega \subseteq M^m$ , and  $\mathcal{S} = \{S_u : u \in \Omega\}$  is a definable family of subsets of  $M^n$  with the FIP. The following are equivalent.*

- (1)  $\mathcal{S}$  can be extended to a definable type in  $S_n(M)$ .
- (2) There exists a definable downward directed family  $\mathcal{F}$  that is finer than  $\mathcal{S}$ .
- (3) There exists a definable curve  $\gamma : (0, \infty) \rightarrow \cup \mathcal{S}$  such that, for every  $u \in \Omega$ ,  $\gamma(t) \in S_u$  for sufficiently small  $t > 0$ .

*Proof.* It is easy to show that (3) implies (1) and (2). Namely, let  $\gamma$  be as in (3), and consider the definable downward directed family  $\{\gamma[(0, t)] : t > 0\}$ . This family is finer than  $\mathcal{S}$ , from which we conclude (2). Moreover, to derive (1), let  $\mathcal{M}(\xi)$  be an elementary extension of  $\mathcal{M}$  by an infinitesimal element. Then the element  $\gamma(\xi) \in M(\xi)^n$  belongs in the interpretation in  $\mathcal{M}(\xi)$  of  $S_u$  for every  $u \in \Omega$ , and so the definable  $n$ -type  $\text{tp}(\gamma(\xi)/M)$  extends  $\mathcal{S}$ . Moreover (2) $\Rightarrow$ (3) is Corollary 4.3.4. It therefore remains to show that (1) implies (2) or (3). We show that (1) implies (3). The main tool will be the completeness of the theory of tame pairs.

Suppose that  $\mathcal{S}$  can be extended to a definable  $n$ -type  $\mathcal{S}'$ . If  $\mathcal{S}'$  is realised in  $\mathcal{M}$  it suffices to let  $\gamma$  be a constant curve mapping to the realisation of  $\mathcal{S}'$ . Hence we assume that  $\mathcal{S}'$  is not realised. Let  $\mathcal{N} = (N, <, \dots)$  be a (necessarily proper) elementary extension of  $\mathcal{M}$  such that there exists an element  $c \in N^n$  that realizes  $\mathcal{S}'$ , and let  $\mathcal{M}(c)$  be the prime model over  $M \cup \{c\}$ . Then  $\mathcal{M} \preceq \mathcal{M}(c)$ . By definable choice  $\mathcal{M}(c)$  is the definable closure of  $M \cup \{c\}$  in  $\mathcal{N}$ , in particular any definable set in  $\mathcal{M}(c)$  is  $M \cup \{c\}$ -definable. Since  $\mathcal{S}' = \text{tp}(c/M)$  is definable it follows that  $\mathcal{M}(c)$  is a tame extension of  $\mathcal{M}$ . Since  $\mathcal{S} \subseteq \mathcal{S}'$  we have that, for every  $u \in \Omega$ , the element  $c$  belongs in the interpretation of  $S_u$  in  $\mathcal{M}(c)$ . Now let  $\psi(u)$  be a formula defining  $\Omega$  (by adding constants to the language if necessary we may assume that  $\Omega$  is 0-definable) and let  $\phi(u, v)$  be a formula such that, for every  $u \in \Omega$ ,  $S_u = \phi(u, M^n)$ , then  $(\mathcal{M}(c), \mathcal{M})$  satisfies the following sentence in the language of tame pairs:

$$\exists v \forall u ((\psi(u) \cap u \in M^m) \rightarrow \phi(u, v)).$$

Let  $\mathcal{M}(\xi)$  be an elementary extension of  $\mathcal{M}$  generated by an infinitesimal element with respect to  $\mathcal{M}$ . By the completeness of the theory of tame pairs, there exists  $\gamma(\xi) \in \mathcal{M}(\xi)$  such that  $\mathcal{M}(\xi) \models \phi(u, \gamma(\xi))$  for every  $u \in \Omega$ , where  $\gamma$  is a definable curve. It follows that, for every  $u \in \Omega$ ,  $\gamma(t) \in S_u$  for all  $t > 0$  small enough in  $M$ . Hence we conclude (3).  $\square$

The equivalence (1) $\Leftrightarrow$ (2) in Theorem 4.4.2 establishes a connection between definable types and downward directed families of sets in o-minimal expansions of ordered fields. We extend this connection further, proving that this equivalence holds in all o-minimal structures, in Chapter 5 (Theorem 5.2.11).

**Definition 4.4.3.** We call a partial type  $p(v)$  a  $\phi$ -type, for some formula  $\phi(u, v)$ , if there exists a set of parameters  $\Omega \subseteq M^{l(u)}$  such that  $p(v) = \{\phi(u, v) : u \in \Omega\}$ .

Given the terminology we give widespread use in Chapter 5, a  $\phi$ -type is a uniform family of sets with the FIP. A definable  $\phi$ -type is a definable family of sets with the FIP.

**Remark 4.4.4.** As noted in the proof of Theorem 4.4.2, it follows easily from o-minimality that (3) in the theorem implies (1) and (2), and the crux of the result lies in proving that (1) implies (3) and (2) implies (3). These are really two separate results, the former following from the completeness of the theory of tame pairs and the latter from results in previous sections.

Note that Theorem 4.4.2 still holds without the assumption that the index set  $\Omega$  be definable, i.e. it suffices that  $\mathcal{S}$  is a  $\phi$ -type for some formula  $\phi$ . Moreover, only the implication (1) $\Rightarrow$ (3) requires this uniformity assumption. For the other implications it is enough that  $\mathcal{S}$  is a partial type.

The dimension of a type  $p$  is defined to be the minimum dimension among sets in  $p$ . O-minimality implies that every one-dimensional type has a uniform basis. The same is trivially true for zero-dimensional types.

Note that the proof of the implication (3) $\Rightarrow$ (1) in Theorem 4.4.2 involves showing that, by o-minimality, for any definable curve  $\gamma$ , the definable family  $\{\gamma[(0, t)] : t > 0\}$  is a (uniform) basis for a definable type.

The following lemma is easy to prove and follows partly from the proof of Theorem 4.4.2.

**Lemma 4.4.5.** Suppose that  $\mathcal{M}$  expands an ordered field and let be  $\mathcal{S}$  a partial type, then (3) in Theorem 4.4.2 is equivalent to any of the following statements.

- (1)  $\mathcal{S}$  is realized in  $\mathcal{M}(\xi)$ , where  $\xi$  is an element infinitesimal with respect to  $\mathcal{M}$ .
- (2)  $\mathcal{S}$  extends to a definable type of dimension at most one (which has a uniform basis).

By Lemma 4.4.5 (2) it follows that the implications (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3) in Theorem 4.4.2 can be formulated in terms of types as follows.



**Proposition 4.4.6.** *Let  $p$  be a  $\phi$ -type, for some formula  $\phi$ . Suppose that either of the following holds.*

(i)  *$p$  can be extended to a definable type.*

(ii)  *$p$  can be extended to a partial definable type with a uniform basis.*

*Then  $p$  extends to a definable type of dimension at most one (with a uniform basis).*

If we only assume that  $\mathcal{M}$  expands an ordered group, then we may prove a result similar to the equivalence (2) $\Leftrightarrow$ (3) in Theorem 4.4.2, where we substitute (3) with the existence of a definable map  $\gamma : M^{>0} \times M^{>0} \rightarrow \cup \mathcal{S}$  where the definable family of sets  $\{\gamma[(0, s) \times (t, \infty)] : s, t > 0\}$  is finer than  $\mathcal{S}$ . Note that this family is a (uniform) basis for a type of dimension at most two.

From this observation and the implication (2) $\Rightarrow$ (3) in Theorem 4.4.2 we derive the next corollary.

**Corollary 4.4.7.** *Every partial definable type with a uniform basis may be extended to a definable type of dimension at most two. In particular, every definable type with a uniform basis has dimension at most two. If  $\mathcal{M}$  expands an ordered field then every partial definable type with a uniform basis extends to a definable type of dimension at most one, and every definable type with a uniform basis has dimension at most one.*

**Remark 4.4.8.** The bounds in Corollary 4.4.7 are tight. In particular if  $\mathcal{M}$  does not expand an ordered field then the family  $\{(0, s) \times (t, \infty) : s, t > 0\}$  is a basis for a (two-dimensional) type. To see this recall Fact 4.3.1 and note that, if  $\mathcal{M}$  does not have a pole, then any cell  $(f, g)$  that intersects every box of the form  $(0, s) \times (t, \infty)$  must satisfy that  $f = +\infty$  and  $\lim_{t \rightarrow 0^+} g(t) < \infty$ , and so it contains one such box. By cell decomposition it follows that, for every definable set  $X \subseteq M^2$ , there are  $s_X, t_X > 0$  such that  $(0, s_X) \times (t_X, \infty)$  is either contained in or disjoint from  $X$ .

From the fact that all types in an o-minimal expansion of  $(\mathbb{R}, <)$  are definable (see Theorem 2.1.8 and recall that all extensions of  $(\mathbb{R}, <)$  are tame) we may derive, using the implication (1) $\Rightarrow$ (3) in Theorem 4.4.2, the next corollary for o-minimal expansions of ordered archimedean fields.

**Corollary 4.4.9.** *Suppose that  $\mathcal{M}$  expands an ordered field. The following are equivalent:*

- (1)  $\mathcal{M}$  is archimedean.
- (2) If  $\mathcal{S}$  is a definable family of sets with the FIP, then there exists a definable curve  $\gamma : (0, \infty) \rightarrow \cup \mathcal{S}$  such that, for every  $S \in \mathcal{S}$ ,  $\gamma(t) \in S$  for sufficiently small  $t > 0$ .
- (3) Every definable family  $\mathcal{S}$  with the FIP can be extended to a definable type.

*Proof.* The implication (2) $\Rightarrow$ (3) follows from o-minimality. We prove (3) $\Rightarrow$ (1) and (1) $\Rightarrow$ (2).

To show (3) $\Rightarrow$ (1), suppose that  $\mathcal{M}$  is nonarchimedean and let  $r \in M$  denote an infinitesimal with respect to 1. Consider the definable family of sets  $\mathcal{S} = \{(0, 1) \setminus (a - r, a + r) : a \in [0, 1]\}$ . Because  $r$  is infinitesimal this family has the FIP. It cannot however be extended to a definable type  $p(v)$ , since in that case the set  $\{t : (t \leq v) \in p(v)\}$  would have a supremum  $s \in [0, 1]$ , contradicting  $((v < s - r) \vee (s + r < v)) \in p(v)$ .

To show (1) $\Rightarrow$ (2), suppose that  $\mathcal{M}$  expands an archimedean field and let  $\mathcal{S} = \{S_u \subseteq M^n : u \in \Omega\}$  be a definable family with the FIP. The result amounts to the following claim: that, for some  $(1+n+m)$ -formula  $\phi(t, v, w)$ , there are parameters  $b \in M^m$  such that  $\phi(t, v, b)$  defines a curve  $\gamma$  in  $M^n$  (given by  $t \mapsto v$ ) with the property that, for all  $u \in \Omega$ ,  $\gamma(t) \in S_u$  whenever  $t > 0$  is small enough. So, in order to complete the proof, it suffices to show this in an elementary extension of  $\mathcal{M}$ .

Laskowski and Steinhorn [30] proved that any o-minimal expansion of an archimedean ordered group is elementarily embeddable into an o-minimal expansion of the additive ordered group of real numbers. Moreover note that the fact that the family  $\mathcal{S}$  has the FIP is witnessed by countably many sentences in the elementary diagram of  $\mathcal{M}$ , and so the same property holds for the family  $\mathcal{S}$  interpreted in any elementary extension of  $\mathcal{M}$ . Therefore, by Laskowski-Steinhorn [30], we may assume that  $\mathcal{M}$  expands the ordered additive group of real numbers.

By Dedekind completeness of the reals, every elementary extension of  $\mathcal{M}$  is tame, and thus by the Marker-Steinhorn Theorem (Theorem 2.1.8) every type over  $M$  is definable. In particular any expansion of  $\mathcal{S}$  to a type over  $M$  is definable. The corollary then follows from Theorem 4.4.2. □

## 4.5 Definable first countability

In this section we use results from Sections 4.2 and 4.3 to prove facts about definable topological spaces. Note that a basis of neighborhoods of a point in a topological space is a downward directed family of sets. If the topology is definable then this basis can be chosen definable too.

As usual throughout we omit explicit references to a topology whenever this topology is clear from context.

When making the assumption that  $\mathcal{M}$  expands a field we will always understand convergence of a curve  $\gamma$  to mean convergence as  $t \rightarrow 0$ .

Recall the classical definitions of the *density*,  $d(X)$ , and *character*,  $\chi(X)$ , of a topological space  $(X, \tau)$ , namely

$$d(X) = \min\{|D| : D \subseteq X \text{ is dense in } X\}$$

and

$$\chi(X) = \sup_{x \in X} \min\{|\mathcal{B}| : \mathcal{B} \text{ a basis of neighborhoods of } x\}.$$

In particular  $X$  is separable if and only if  $d(X) \leq \aleph_0$ , and first countable if and only if  $\chi(X) \leq \aleph_0$ . Given a definable set  $Y$  we write  $d(Y)$  to denote the density of  $Y$  with the euclidean topology.

The first result in this section is an observation that follows only from o-minimality, that is, we do not require the assumption that  $\mathcal{M}$  expands an ordered group. It implies that any definable topological space in an o-minimal expansion of  $(\mathbb{R}, <)$  is first countable.

**Proposition 4.5.1.** *Every definable downward directed set admits a downward cofinal subset of cardinality at most  $d(M)$ .*

*It follows that every definable topological space  $(X, \tau)$  satisfies  $\chi(X) \leq d(M)$ . In particular, if  $(M, <)$  is separable, then any definable topological space is first countable.*

Note that this proposition cannot be improved by writing second countable in place of first countable, since the discrete topology on  $M$  is definable and not second countable whenever  $M$  is uncountable.

*Proof of Proposition 4.5.1.* First we recall some facts of o-minimality. Let  $D_M \subseteq M$  be a set dense in  $M$  and let  $C \subseteq M^n$  be a cell of dimension  $m$ . There exists a projection  $\pi : C \rightarrow M^m$  that is a homeomorphism onto an open cell  $C'$ . If  $m = 0$  then  $d(C) = 1$ . Otherwise the set  $\pi^{-1}(D_M^m \cap C')$ , where  $D_M^m = D_M \times \cdots \times D_M$ , is clearly dense in  $C$ . By o-minimal cell decomposition, it follows that any definable set  $B$  satisfies  $d(B) \leq d(M)$ . Secondly recall that, by the frontier dimension inequality, for any definable set  $B$  and definable subset  $B'$ , if  $\dim B = \dim B'$  then  $B'$  has interior in  $B$ .

Now set  $\lambda = d(M)$ . Let  $(\Omega, \preceq)$  be a definable downward directed set. For any  $u \in \Omega$ , let  $S_u = \{v \in \Omega : v \preceq u\}$ . By Remark 4.1.3 the definable family  $\mathcal{S} = \{S_u : u \in \Omega\}$  is downward directed. We fix  $S_* \in \mathcal{S}$  such that  $\dim S_* = \min\{\dim S : S \in \mathcal{S}\}$ , and let  $D$  be a dense subset of  $S_*$  of cardinality at most  $\lambda$ . We claim that  $D$  is downward cofinal in  $\Omega$ . This is because, for every  $S_u \in \mathcal{S}$ , there exists  $S' \in \mathcal{S}$  such that  $S' \subseteq S_u \cap S_*$ , and so  $\dim(S_u \cap S_*) = \dim(S_*)$ , and in particular  $S_u \cap S_*$  has interior in  $S_*$ . This means that, for every  $u \in \Omega$ , there exists some  $v \in D$  such that  $v \in S_u \cap S_*$ , and so  $v \preceq u$ .

Let  $(X, \tau)$  be a definable topological space. For any given  $x \in X$ , let  $\mathcal{B}(x) = \{A_u : u \in \Omega_x\}$  be a definable basis of neighborhoods of  $x$ . Let  $(\Omega_x, \preceq_x)$  denote the definable downward directed set given  $u \preceq_x v \Leftrightarrow A_u \subseteq A_v$ . Note that, if  $D$  is a downward cofinal subset of  $\Omega_x$ , then the family  $\{A_u : u \in D\}$  is still a basis of neighborhoods of  $x$  in  $(X, \tau)$ . This completes the proof of the proposition.  $\square$

We now make use of Theorem 4.2.2 and Corollary 4.3.4 to prove the main theorem of this section. It might strike the reader as surprising that we name this theorem “definable first countability”, as opposed to using that label for Proposition 4.5.1. The reasons for this will be explained later in the section, but in short it is due to the fact that a consequence (Lemma 4.5.4) of Theorem 4.5.2 is that, whenever  $\mathcal{M}$  expands an ordered field, any definable topological space admits definable curve selection.

**Theorem 4.5.2** (Definable first countability). *Let  $(X, \tau)$  be a definable topological space with definable basis  $\mathcal{B}$ .*

(1) For every  $x \in X$ , there exists a definable basis of neighborhoods of  $x$  of the form  $\{A_{(s,t)} : s, t > 0\} \subseteq \mathcal{B}$  satisfying that, for any neighborhood  $A$  of  $x$ , there exist  $s_A, t_A > 0$  such that  $A_{(s,t)} \subseteq A$  whenever  $(s, t) \trianglelefteq (s_A, t_A)$ .

(2) Suppose that  $\mathcal{M}$  expands an ordered field. Then, for every  $x \in X$ , there exists a definable basis of neighborhoods of  $x$  of the form  $\{A_t : t > 0\} \subseteq \mathcal{B}$  satisfying that, for any neighborhood  $A$  of  $x$ , there exists  $t_A > 0$  such that  $A_t \subseteq A$  whenever  $0 < t \leq t_A$ .

*Proof.* Let  $(X, \tau)$  be a definable topological space with definable basis  $\mathcal{B} = \{A_u : u \in \Omega\}$ . We begin by proving (1).

Given  $x \in X$  let  $\Omega_x = \{u \in \Omega : x \in A_u\}$ . The family  $\{A_u : u \in \Omega_x\} \subseteq \mathcal{B}$  is a definable basis of neighborhoods of  $x$  that induces a definable preorder  $\preceq_{\mathcal{B}}$  by inclusion (see Remark 4.1.3(i)) making  $(\Omega_x, \preceq_{\mathcal{B}})$  into a definable downward directed set. By Theorem 4.2.2, let  $\gamma : (M^{>0} \times M^{>0}, \trianglelefteq) \rightarrow (\Omega_x, \preceq_{\mathcal{B}})$  be a definable downward cofinal map. The family  $\{A_{\gamma(s,t)} : s, t > 0\}$  has the desired properties.

The proof of (2) follows analogously from Corollary 4.3.4. □

We denote the neighborhood bases described in (1) and (2) in the above theorem *cofinal bases of neighborhoods of  $x$  in  $(X, \tau)$* . By using Corollaries 4.2.13 and 4.3.4, one may show that these may be chosen uniformly on  $x \in X$ .

The next corollary is a refinement of Theorem 4.5.2. It shows in particular that the cofinal bases of neighborhoods described in (2) may be assumed to be nested.

**Corollary 4.5.3.** *Let  $(X, \tau)$  be a definable topological space.*

(1) For every  $x \in X$ , there exists a definable basis of open neighborhoods of  $x$  of the form  $\{A_{(s,t)} : s, t > 0\}$  satisfying  $A_{(s',t')} \subseteq A_{(s,t)}$  whenever  $(s', t') \trianglelefteq (s, t)$ .

(2) Suppose that  $\mathcal{M}$  expands an ordered field. Then, for any  $x \in X$ , there exists a definable basis of open neighborhoods of  $x$  of the form  $\{A_t : t > 0\}$  satisfying  $A_s \subseteq A_t$  whenever  $0 < s < t$ .

*Proof.* We prove (2), and then sketch how statement (1) follows in a similar fashion.

Let  $(X, \tau)$  be a definable topological space and suppose that  $\mathcal{M}$  expands an ordered field. Let  $x \in X$  and  $\mathcal{A} = \{A_t : t > 0\}$  be a cofinal basis of neighbourhoods of  $x$  as given by Theorem 4.5.2 (2). Let  $f : (0, \infty) \rightarrow (0, \infty)$  be the function defined as follows. For every  $t > 0$ ,  $f(t) = \sup\{s < 1 : A_{s'} \subseteq A_t \text{ for every } 0 < s' < s\}$ . By definition of  $\mathcal{A}$  and o-minimality, this function is well defined and definable. By o-minimality, let  $(0, r_0]$  be an interval on which  $f$  is continuous. Note that the family  $\{A_t : 0 < t \leq r_0\}$  is still a basis of neighborhoods of  $x$ . By continuity, for every  $0 < r < r_0$ ,  $f$  reaches its minimum in  $[r, r_0]$ , so there exists  $t_r > 0$  such that  $A_{t_r} \subseteq \bigcap_{r \leq t \leq r_0} A_t$ . In particular the set  $\bigcap_{r \leq t \leq r_0} A_t$  remains a neighborhood of  $x$ . For every  $t > 0$ , consider the definable set

$$A'_t = \begin{cases} \bigcap_{t \leq t' \leq r_0} A_{t'} & \text{if } t < r_0; \\ A_{r_0} & \text{otherwise.} \end{cases}$$

The family  $\{\text{int}_\tau(A'_t) : t > 0\}$  is a definable basis of open neighborhoods of  $x$  such that, for every  $0 < s < t$ ,  $\text{int}_\tau(A'_s) \subseteq \text{int}_\tau(A'_t)$ .

To prove (1) note first that, if  $\mathcal{M}$  expands an ordered field, then we may use the construction in (2) taking  $A_{(s,t)} = A_s$ . Suppose that  $\mathcal{M}$  does not expand an ordered field (equivalently, by Fact 4.3.1,  $\mathcal{M}$  does not have a pole). Let  $x \in X$  and let  $\{A_{(s,t)} : s, t > 0\}$  be a basis of neighborhoods of  $x$  as given by Theorem 4.5.2 (1). Using definable choice, let  $f$  be a definable map on  $M^{>0} \times M^{>0}$  such that  $A_{(s',t')} \subseteq A_{(s,t)}$  for every  $s', t' > 0$  with  $(s', t') \trianglelefteq f(s, t)$ . By o-minimality let  $C$  be a cell, cofinal in  $(M^{>0} \times M^{>0}, \trianglelefteq)$ , where  $f$  is continuous. Since  $\mathcal{M}$  does not have a pole there exists some box  $(0, r_0] \times [r_1, \infty)$  contained in  $C$  (see Remark 4.4.8). Following the approach to proving (2), let  $A'_{(s,t)} = \bigcap\{A_{(s',t')} : (s, t) \trianglelefteq (s', t') \trianglelefteq (r_0, r_1)\}$  if  $(s, t) \trianglelefteq (r_0, r_1)$ , and  $A'_{(s,t)} = A_{(r_0, r_1)}$  otherwise. Our desired basis is given by  $\{\text{int}_\tau(A'_{(s,t)}) : s, t > 0\}$ .  $\square$

Recall that in general topology a *net* is a map from a (generally upward) directed set into a topological space for which there is a notion of convergence. The utility of nets lies in that they completely capture the topology in the following sense: the closure of a subset  $Y$  of a topological space  $(X, \tau)$  is the set of all limit points of nets in  $Y$ . It follows that nets encode the set of points on which a given function between topological spaces is continuous.

It seems natural to define a *definable net* to be a definable map from a definable directed set (which we can choose to be downward definable directed set for convenience) into a definable topological space. Indeed, applying definable choice one may show that definable nets defined in this way have properties that are equivalent to those of nets in general topology, when considering definable subsets of definable topological spaces and definable functions between definable topological spaces.

Theorem 4.5.2 suggests that, for all practical purposes, it is enough to consider definable nets to be those whose domain is  $(M^{>0} \times M^{>0}, \trianglelefteq)$ , if  $\mathcal{M}$  expands an ordered group but does not expand an ordered field, and to simply consider them as definable curves if  $\mathcal{M}$  does expand an ordered field (see Lemma 4.5.4 and Corollary 4.5.5 below). This is analogous to the way in which, in first countable spaces, for all purposes nets can be taken to be sequences. It is from this last observation, namely that, whenever  $\mathcal{M}$  expands an ordered field, definable curves take the role in definable topological spaces of sequences in first countable spaces, that the motivation to label Theorem 4.5.2 “definable first countability” arose.

This argument of course implicitly relates definable curves in the definable o-minimal setting to sequences in general topology. This identification however is far from new in the o-minimal setting. It is mentioned for example by van den Dries in [17], page 93, with respect to the euclidean topology in the context of o-minimal expansion of ordered groups, and implicitly noted by Thomas in [48] when introducing the notions of *definable Cauchy curves* and closely related *definable completeness* in the context of certain topological spaces of definable functions where the underlying o-minimal structure expands an ordered field. The same implicit identification can be seen in the definition of definable compactness in [37], which is given in terms of convergence of definable curves. The motivation for this correspondence lies precisely in that, in the settings being considered by the authors, definable curves display properties similar to those of sequences in the corresponding setting of classical topology.

We conclude this section with results illustrating that, whenever  $\mathcal{M}$  expands an ordered field, definable curves indeed take the role of definable nets, displaying properties similar to those of sequences in first countable topological spaces. In particular we show that definable topological spaces admit definable curve selection, and consequently, by Proposition 2.2.14,

continuity of definable functions can be characterized in terms of convergence of definable curves.

**Lemma 4.5.4** (Definable curve selection). *Suppose that  $\mathcal{M}$  expands an ordered field. Any definable topological space  $(X, \tau)$  has definable curve selection, that is, if  $Y \subseteq X$  is a definable set and  $x \in X$ , then  $x$  belongs to the closure of  $Y$  if and only if there exists a definable curve  $\gamma$  in  $Y$  that converges to  $x$ .*

*Proof.* Let  $(X, \tau)$  be a definable topological space and  $x \in X$ . Let  $Y \subseteq X$  be a definable set and suppose that  $x \in cl_\tau(Y)$ . If  $\mathcal{M}$  expands an ordered field there exists, by Theorem 4.5.2, a definable basis of neighborhoods of  $x$ ,  $\{A_t : t > 0\}$ , such that, for every neighborhood  $A$  of  $x$ ,  $A_t$  is a subset of  $A$  for all  $t > 0$  small enough.

Given one such definable basis of neighborhoods, consider by definable choice a definable curve  $\gamma$  such that, for every  $t > 0$ ,  $\gamma(t) \in A_t \cap Y$ . This curve clearly lies in  $Y$  and converges in  $(X, \tau)$  to  $x$ .

Conversely it follows readily from the definition of curve convergence that, if there exists a curve in  $Y$  converging in  $(X, \tau)$  to  $x \in X$ , then  $x \in cl_\tau(Y)$ . □

The following follows directly from Lemma 4.5.4 and Proposition 2.2.14.

**Corollary 4.5.5.** *Suppose that  $\mathcal{M}$  expands an ordered field. Let  $(X, \tau)$  and  $(Y, \mu)$  be definable topological spaces. Let  $f : (X, \tau) \rightarrow (Y, \mu)$  be a definable map. Then, for any  $x \in X$ ,  $f$  is continuous at  $x$  if and only if, for every definable curve  $\gamma$  in  $X$ , if  $\gamma$   $\tau$ -converges to  $x$  then  $f \circ \gamma$   $\mu$ -converges to  $f(x)$ .*



## 5. DEFINABLE COMPACTNESS

### Introduction

In this chapter we seek to characterize a notion of definable compactness for definable topological spaces in o-minimal structures, in particular in terms of intersecting definable families of closed sets. Our definition of definable compactness is that every definable downward directed family of closed sets has nonempty intersection (Definition 5.4.1)

Inspired by the results in Chapter 4, we begin by studying the relationship between types and definable downward directed families of sets. We are able to prove that any such family extends to a definable type, and conversely that any definable family of sets that extends to a definable type also extends to one with a definable basis of cells (Theorem 5.2.11). In particular this is a strong density result for types with a definable basis. This approach to studying types seems to be new. It allows us to translate our questions about families of closed sets in definably compact topologies into questions about intersecting definable families of sets in general in connection with definable types.

We study definable families of sets with the  $(p, q)$ -property, i.e. the property that, for every  $p$  sets in the family, some  $q$  intersect. In Theorem 5.3.9 we give an elementary proof within o-minimality of the fact that, if a definable family of subsets of  $M^n$  has the  $(p, n + 1)$ -property, for any  $p > n$ , then it admits a finite covering of subfamilies, each of which extends to a definable type (we call this having a finite tame transversal). We also derive a similar result in terms of VC-codensity (Corollary 5.3.11).

We observe how Theorem 5.3.9 is equivalent to the known fact that, in an o-minimal structure, a formula does not divide (equivalently does not fork) over a model  $M$  if and only if it extends to an  $M$ -definable type. This equivalence is known in a large class of dp-minimal theories [47]. We prove it within o-minimality for dividing over any set (not just a model); see Theorem 5.3.19.

Finally, we prove our characterization of definable compactness (Theorem 5.4.9). Using our density result on types we are able to show that a space is definably compact if and only if every definable family of closed sets that extends to a definable type has nonempty intersection (specialization-compactness). Using our  $(p, q)$ -theorem and corollary we derive

that, in a definably compact space, any definable family of closed sets with the  $(p, q)$ -property, for some  $p \geq q$  with  $q$  large enough, has a finite transversal (i.e. there is a finite set that intersects every set in the family). We also establish the relationship between definable compactness and the classical namesake notion in terms of convergence of definable curves (definable curve-compactness) introduced in [37].

Theorem 2.1 in [35] shows that, in an o-minimal theory with definable choice functions, if a non-forking formula over a model  $M$  defines (in a monster model) a closed and bounded set  $C$ , then  $C \cap M^n \neq \emptyset$ . And from this it is derived that every definable family of closed and bounded sets with the finite intersection property has a finite transversal. Our characterization of definable compactness generalizes this result (see Remark 5.4.10) to any formula that defines a definably compact closed set in some  $M$ -definable topology, also dropping the assumption of having choice functions. In the same remark we explain how our work can be used to expand on the definable Helly's Theorem proved in [5] by showing in particular that, in an o-minimal expansion of an ordered field  $\mathcal{M}$ , any family of convex subsets of  $M^n$  with the property that every subfamily of size  $n + 1$  has nonempty intersection extends to a definable type.

Our focus throughout this chapter is on definable families of sets, and our proofs are mostly elementary, in that they rely solely on o-minimal cell decomposition.

In Section 5.1 we introduce the necessary preliminaries. In Section 5.2 we study o-minimal types and prove Theorem 5.2.11. Section 5.3 includes our work on definable families of sets with the  $(p, q)$ -property; we prove Theorem 5.3.9 and Corollary 5.3.11. In Section 5.3.1 we note the connection between our work and known properties of forking in o-minimal theories. Finally, in Section 5.4 we prove our characterization of definable compactness, Theorem 5.4.9.

Using the approach of various proofs in this chapter we include in Appendix 5.A a shortened proof of the Marker-Steinhorn Theorem (Theorem 2.1.8).

This chapter is based on a paper by the author in preparation [1].

## 5.1 Preliminaries

Recall the definition and conventions on types from Chapter 2 (Section 2.1.4). In particular we treat partial types indistinctly as consistent sets of formulas and as families of definable sets with the finite intersection property (FIP). Throughout, unless otherwise specified, all types are global (over  $M$ ) and complete. Then  $S_n(M)$  denotes the set of  $n$ -types. In this chapter we make ample use of the notions of definable type and uniform (type) basis (Definition 2.1.7).

Recall that  $Def(M^n)$  denotes the algebra of definable subsets of  $M^n$ . Moreover  $\pi : M^{n+1} \rightarrow M^n$  denotes the projection to the first  $n$  coordinates. For a family  $\mathcal{S} \subseteq Def(M^{n+1})$ , recall that  $\pi(\mathcal{S}) = \{\pi(S) : S \in \mathcal{S}\}$ .

Let  $A \subseteq M$ . Note that, if  $p$  is an  $A$ -definable type, then the projection  $\pi(p)$  is also an  $A$ -definable type.

**Definition 5.1.1.** *A family of sets  $\mathcal{S} \subseteq Def(M^n)$  is uniform<sup>1</sup> if there is a formula  $\varphi(u, v)$  and a set  $\Omega \subseteq M^{l(u)}$  such that  $\mathcal{S} = \{\varphi(u, M^{l(v)}) : u \in \Omega\}$ . We call  $\Omega$  the index set.*

Note that a uniform family  $\mathcal{S} = \{\varphi(u, M^{l(v)}) : u \in \Omega\}$  is definable whenever  $\Omega$  can be chosen definable. For example if  $\mathcal{M}$  denotes the real algebraic numbers then the family of intervals  $(s, t)$  for  $s < \pi < t$  is uniform but not definable.

### 5.1.1 Preorders induced by types

Recall that a preorder is a reflexive transitive relation. A preordered set  $(X, \preceq)$  is a set  $X$  together with a preorder  $\preceq$  on it. It is definable if the preorder is definable (Definition 4.1.1). A preorder  $\preceq$  on a set  $X$  induces an equivalence relation  $\sim$  on  $X$  given by  $x \sim y$  if and only if  $x \preceq y$  and  $y \preceq x$ . We use notation  $x \prec y$  to mean  $x \preceq y$  and  $x \not\sim y$ . Given a set  $Y \subseteq X$  let  $x \preceq Y$  (respectively  $x \prec Y$ ) mean  $x \preceq y$  (respectively  $x \prec y$ ) for every  $y \in Y$ . For every  $x, y \in X$  let  $[x, y]_{\preceq} = \{z \in X : x \preceq z \preceq y\}$  and  $(x, y)_{\preceq} = \{z \in X : x \prec z \prec y\}$ .

Let  $p$  be an  $n$ -type and  $\mathcal{G}$  be the collection of all definable functions  $f : M^n \rightarrow M_{\pm\infty}$  such that  $dom(f) \in p$ . Then  $p$  induces a total preorder  $\preceq$  on  $\mathcal{G}$  given by  $f \preceq g$  if and only

<sup>1</sup>↑We choose this terminology in part to justify the description of a “uniform topology” among types in Remark 5.2.12, in connection with the uniform convergence topology in function spaces

if  $\{u \in M^n : f(u) \leq g(u)\} \in p$ . In other words,  $f \preceq g$  when  $f(\xi) \leq g(\xi)$  for some (every) realization  $\xi$  of  $p$ .

Let  $\mathcal{F} = \{f_x : x \in X\} \subseteq \mathcal{G}$  be a uniform family. Without loss of clarity we will often abuse notation and refer to  $\preceq$  too as the total preorder on the index set  $X$  given by  $x \preceq y$  if and only if  $f_x \preceq f_y$ . Note that, if  $\mathcal{F}$  and  $p$  are  $A$ -definable, then  $\preceq$  on  $X$  is  $A$ -definable too.

### 5.1.2 Transversals and intersection properties

For  $\mathcal{S}$  a family of sets and  $F$  a set, let  $\mathcal{S} \cap F = \{S \cap F : S \in \mathcal{S}\}$ . The next two definitions refer to classical terminology in combinatorics.

**Definition 5.1.2.** *Given a family of sets  $\mathcal{S}$  and a set  $F$ , we say that  $F$  is a transversal of  $\mathcal{S}$  if it intersects every set in  $\mathcal{S}$ , i.e. if  $\mathcal{S} \cap F$  does not contain the empty set.*

In this chapter we are interested in the property that a definable family of sets has a finite transversal, and in a similar property in terms of types that we introduce in Section 5.3 (Definition 5.3.2).

**Definition 5.1.3.** *Let  $\mathcal{S}$  be a family of sets. We say that  $\mathcal{S}$  is  $n$ -consistent if any subfamily of cardinality at most  $n$  has nonempty intersection.*

*We say that  $\mathcal{S}$  is  $n$ -inconsistent if every subfamily of cardinality  $n$  has empty intersection.*

*We say that  $\mathcal{S}$  has the  $(p, q)$ -property, for  $p \geq q > 0$ , if every subfamily  $\mathcal{S}' \subseteq \mathcal{S}$  of size  $p$  contains a subfamily of size  $q$  with nonempty intersection.*

Note that, for  $p$  large with respect to  $q$ , the  $(p, q)$ -property is a rather weak intersection property. A family of sets  $\mathcal{S}$  does not have the  $(p, q)$ -property if and only if there exists a subfamily of  $\mathcal{S}$  of size  $p$  that is  $q$ -inconsistent.

As an example of a fact involving transversals we include the following proposition, whose proof (which can be obtained using Mirsky's theorem, [33], Theorem 2) we leave to the reader.

**Proposition 5.1.4.** *Let  $\mathcal{S} \subseteq \mathcal{P}(M)$  be a family of closed intervals. Let  $k \geq 1$  be the maximum such that there exists  $k$  pairwise disjoint sets in  $\mathcal{S}$ . Then  $\mathcal{S}$  has a transversal of size  $k$ .*

### 5.1.3 VC theory

In this subsection we present the basic notions of VC theory, and state the Alon-Kleitman-Matoušek  $(p, q)$ -theorem, as well as a known result bounding the VC-codensity of formulas in o-minimal structures. These results have applications in Section 5.3. The following brief introduction serves as context for them. Only the notion of VC-codensity will appear in the proofs in this chapter, always in reference to the theorems.

A pair  $(X, \mathcal{S})$ , where  $X$  is a set and  $\mathcal{S}$  is a family of subsets of  $X$ , is called a *set system*. For a set  $F \subseteq X$  we say that  $\mathcal{S}$  *shatters*  $F$  if  $\mathcal{S} \cap F = \mathcal{P}(F)$ . The *VC-dimension* of  $\mathcal{S}$ , denoted by  $\text{VC}(\mathcal{S})$ , is the maximum cardinality of a finite set  $F$  such that  $\mathcal{S}$  shatters  $F$ , if one such set exists. Otherwise we write  $\text{VC}(\mathcal{S}) = \infty$ . The *shatter function*  $\pi_{\mathcal{S}} : \omega \rightarrow \omega$  is defined by  $\pi_{\mathcal{S}}(n) = \max\{|\mathcal{S} \cap F| : F \subseteq X, |F| = n\}$ . So the VC-dimension of  $\mathcal{S}$  is the largest  $n$  such that  $\pi_{\mathcal{S}}(n) = 2^n$ .

Suppose that  $\mathcal{S}$  has VC-dimension  $k < \infty$ . The Sauer-Shelah Lemma [45] states then that  $\pi_{\mathcal{S}} = O(n^k)$  (that is,  $\pi_{\mathcal{S}}(n)/n^k$  is bounded at infinity). The *VC-density* of  $\mathcal{S}$ , denoted by  $\text{vc}(\mathcal{S})$ , is the infimum over all real numbers  $r \geq 0$  such that  $\pi_{\mathcal{S}} = O(n^r)$ . Hence  $\text{vc}(\mathcal{S}) \leq \text{VC}(\mathcal{S})$ .

The *dual set system* of  $\mathcal{S}$  is the set system  $(X^*, \mathcal{S}^*)$ , where  $X^* = \mathcal{S}$  and  $\mathcal{S}^* = \{\mathcal{S}_x : x \in X\}$  where  $\mathcal{S}_x = \{S \in \mathcal{S} : x \in S\}$ . Hence  $\mathcal{S}^*$  shatters  $\mathcal{S}' \subseteq X^*$  if and only if every field in the Venn diagram induced by  $\mathcal{S}'$  on  $X$  contains at least one point. The *dual shatter function* of  $\mathcal{S}$  is  $\pi_{\mathcal{S}}^* = \pi_{\mathcal{S}^*}$ . Similarly there is the *dual VC-dimension* of  $\mathcal{S}$ ,  $\text{VC}^*(\mathcal{S}) = \text{VC}(\mathcal{S}^*)$ , and *VC-codensity*,  $\text{vc}^*(\mathcal{S}) = \text{vc}(\mathcal{S}^*)$ . These satisfy that  $\text{vc}^*(\mathcal{S}) \leq \text{VC}^*(\mathcal{S}) \leq 2^{\text{VC}(\mathcal{S})+1}$ .

A theory is *NIP* if every uniform family of sets in any model has finite VC-dimension. Every o-minimal theory is NIP.

For convenience we state Matoušek's theorem in terms of VC-codensity. For a finer statement see [32].

**Theorem 5.1.5** (Alon-Kleitman-Matoušek  $(p, q)$ -theorem). *Let  $p \geq q > 0$  be natural numbers and let  $(X, \mathcal{S})$  be a set system such that  $\text{vc}^*(\mathcal{S}) < q$ . Then there is a natural number  $n$  such that, for every finite subfamily  $\mathcal{F} \subseteq \mathcal{S}$ , if  $\mathcal{F}$  has the  $(p, q)$ -property then it has a transversal of size at most  $n$ .*

The following corollary will be useful in Section 5.3. It is a reformulation of the main result for weakly o-minimal structures (a class that contains o-minimal structures) in [4] by Aschenbrenner, Dolich, Haskell, Macpherson and Starchenko. It was previously proved for o-minimal structures by Wilkie (unpublished) and Johnson-Laskowski [26], and for o-minimal expansions of the field of reals by Karpinski-Macintyre [28].

**Theorem 5.1.6.** *Let  $\mathcal{S} \subseteq \text{Def}(M^n)$  be a uniform family of sets. Then  $vc^*(\mathcal{S}) \leq n$ .*

Since throughout this chapter  $p$  and  $q$  are employed as standard terminology for types, in the subsequent sections we address the  $(p, q)$ -property in terms of  $m$  and  $n$ , e.g. the  $(m, n)$ -property.

## 5.2 Types

This section contains results about o-minimal types that will be used later on. Our main result is Theorem 5.2.11, which shows that types with a uniform basis are dense in a rather strong sense among all types. The content of this section was motivated by Theorem 4.4.2 in Chapter 4, which establishes a connection between definable types and definable downward directed families of sets in o-minimal expansions of ordered fields.

We start by noting how results in Chapter 4 imply that there exist types without a uniform basis. In fact this already fails among definable types in any o-minimal expansion of  $(\mathbb{R}, +, <)$ .

**Remark 5.2.1.** Recall that the dimension of a type is the lowest dimension of a set in it. In Chapter 4 we proved the following two statements (Corollary 4.4.7). If  $\mathcal{M}$  expands an ordered group, then every definable type with a uniform basis is of dimension at most two, and if  $\mathcal{M}$  expands an ordered field then one such type is of dimension at most one. The Marker-Steinhorn Theorem (Theorem 2.1.8) implies that every type in an o-minimal expansion of  $(\mathbb{R}, <)$  is definable. So, in any o-minimal expansion of  $(\mathbb{R}, <, +)$  or  $(\mathbb{R}, <, +, \cdot)$ , any type of sufficiently large dimension lacks a uniform basis. Finally note that, for any  $n$ , any expansion of the partial type  $\{M^n \setminus X : X \in \text{Def}(M^n), \dim X < n\}$  is  $n$ -dimensional.

Nevertheless, we end this section by arguing, using the next two lemmas, that types with a uniform basis are dense in a strong sense among all types, and an analogous density result among definable types.

Recall (Definition 4.4.1) that, given two families  $\mathcal{S}, \mathcal{F} \subseteq \text{Def}(M^n)$ , we say that  $\mathcal{S}$  is finer than  $\mathcal{F}$  if, for every  $F \in \mathcal{F}$ , there is  $S \in \mathcal{S}$  such that  $S \subseteq F$ . That is, if  $\mathcal{S}$  and  $\mathcal{F}$  have the FIP then the filter generated by  $\mathcal{S}$  is finer than the one generated by  $\mathcal{F}$ .

**Definition 5.2.2.** *Let  $\mathcal{S} \subseteq \text{Def}(M^n)$  be downward directed and  $\mathcal{X} \subseteq \text{Def}(M^n)$ . We say that  $\mathcal{S}$  is complete for  $\mathcal{X}$  if, for every  $X \in \mathcal{X}$ , there is  $S \in \mathcal{S}$  such that either  $S \subseteq X$  or  $S \cap X = \emptyset$ . We say that  $\mathcal{S}$  is complete if it is complete for  $\text{Def}(M^n)$ , in other words, if  $\mathcal{S}$  is a type basis for some type.*

We may now proceed with the results in this section. Theorem 5.2.11 will follow from Lemmas 5.2.6 and 5.2.9. The following facts are easy to prove and will be used later on, often without notice. They do not require that the downward directed family  $\mathcal{S}$  be definable.

**Fact 5.2.3.** *Let  $\mathcal{S}$  be a downward directed family and  $X$  be a set such that, for every  $S \in \mathcal{S}$ ,  $S \cap X \neq \emptyset$ . Then the family  $\mathcal{S} \cap X = \{S \cap X : S \in \mathcal{S}\}$  is downward directed.*

**Fact 5.2.4.** *Let  $\mathcal{S}$  be a downward directed family and  $\mathcal{X}$  be a finite partition of a set  $X$ . If  $S \cap X \neq \emptyset$  for every  $S \in \mathcal{S}$  then there exists  $Y \in \mathcal{X}$  such that  $S \cap Y \neq \emptyset$  for every  $S \in \mathcal{S}$ .*

**Fact 5.2.5.** *Let  $\mathcal{S}$  be a definable downward directed family that is complete for  $\mathcal{X}$  a finite definable partition of a set  $X$ . If  $S \cap X \neq \emptyset$  for every  $S \in \mathcal{S}$  then there exists a unique  $Y \in \mathcal{X}$  such that  $S \subseteq Y$  for some  $S \in \mathcal{S}$ .*

**Lemma 5.2.6.** *Let  $\mathcal{S}$  be a uniform downward directed family. There exist a uniform downward directed family of cells  $\mathcal{C}$  that is complete and finer than  $\mathcal{S}$ . If  $\mathcal{S}$  is a definable family then  $\mathcal{C}$  can be chosen definable.*

Before presenting the proof of Lemma 5.2.6 we make the following observation. Our work in Chapter 4 involves showing that, if  $\mathcal{M}$  expands an ordered field, every downward directed definable family  $\mathcal{S}$  admits a finer (complete by o-minimality) family of the form  $\{\gamma[(0, t)] : 0 < t\}$  for some definable curve  $\gamma : (0, \infty) \rightarrow \cup \mathcal{S}$ . Moreover, whenever  $\mathcal{M}$  expands

an ordered group, there exists one such finer family of the form  $\{\gamma[(0, s) \times (t, \infty)] : 0 < s < t\}$  for some definable map  $\gamma : (0, \infty) \times (0, \infty) \rightarrow \bigcup \mathcal{S}$ . As mentioned before, this provides us with information on the maximum dimension of sets in a complete definable downward directed family. In general however the picture is different. In a dense linear order  $(M, <)$ , for any  $n$  the definable nested family of  $n$ -dimensional sets of the form  $\{\langle x_1, \dots, x_n \rangle : a < x_1 < \dots < x_n\}$ , for  $a \in M$ , is complete.

*Proof of Lemma 5.2.6.* Fix  $\mathcal{S} \subseteq \mathcal{P}(M^n)$  a uniform downward directed family. We may assume that  $\bigcap \mathcal{S} = \emptyset$ , since otherwise it suffices to take  $\mathcal{C} = \{\{x\}\}$  for any  $x \in \bigcap \mathcal{C}$ . We prove the lemma by induction on  $n$ . We assume that  $\mathcal{S}$  is definable. When  $\mathcal{S}$  is not definable the base case follows from the fact that by o-minimality every 1-type has a uniform basis of intervals, and the inductive step follows the same proof we present.

**Base case:**  $n = 1$ .

Let  $H = \{t \in M : \exists S \in \mathcal{S}, S \cap (-\infty, t] = \emptyset\}$ . If  $H$  is empty then, by o-minimality, for every  $S \in \mathcal{S}$  there is  $t_S$  such that  $(-\infty, t_S) \subseteq S$ , in which case we may take  $\mathcal{C} = \{(-\infty, t) : t \in M\}$ .

Now suppose that  $H$  is nonempty. Note that  $H$  is an interval in  $M$  (possibly right closed) unbounded from below. Let  $a = \sup H$ .

If  $a = \max H < \infty$  then there exists  $S_a \in \mathcal{B}$  such that  $S_a \cap (-\infty, t] = \emptyset$ . We note that, for any  $S \in \mathcal{S}$ , there exists  $t_S > a$  such that  $(a, t_S) \subseteq S$ , so we may take  $\mathcal{C} = \{(a, t) : t > a\}$ . Otherwise by o-minimality there exists  $S \in \mathcal{S}$  and  $r_S > a$  such that  $(a, r_S) \cap S = \emptyset$ , but then any set  $S' \in \mathcal{S}$  with  $S' \subseteq S_a \cap S$  satisfies that  $(-\infty, r_S) \cap S' = \emptyset$ , contradicting that  $a = \sup H$ .

If  $a \notin H$  then, for every  $S \in \mathcal{S}$ ,  $S \cap (-\infty, a] \neq \emptyset$ . We show that it suffices to take  $\mathcal{C} := \{(t, a) : t < a\}$ . Let  $S_a \in \mathcal{S}$  be such that  $a \notin S_a$ . Suppose towards a contradiction that there exists  $S \in \mathcal{S}$  and  $r_S < a$  such that  $(r_S, a) \cap S = \emptyset$ . Since  $r_S \in H$ , there is  $S' \in \mathcal{S}$  such that  $S' \cap (-\infty, r_S] = \emptyset$ . But then, any  $S'' \in \mathcal{B}$  with  $S'' \subseteq S' \cap S \cap S_a$  satisfies that  $(-\infty, a] \cap S'' = \emptyset$ , contradiction. This completes the proof of the base case.

**Inductive step:**  $n > 1$ .



Suppose that there exists a cell  $C \subseteq M^n$  with  $\dim C = m < n$  such that  $S \cap C \neq \emptyset$  for every  $S \in \mathcal{S}$ . Let  $\pi_C : C \rightarrow M^m$  be a projection that is a homeomorphism onto an open cell. By Fact 5.2.3 the family  $\{\pi_C(S \cap C) : S \in \mathcal{S}\}$  is a definable downward directed family of subsets of  $M^m$ . By inductive hypothesis it admits a finer complete definable downward directed family of cells  $\mathcal{C}'$ . By passing to a subfamily of  $\mathcal{C}'$  if necessary we may assume that every cell in  $\mathcal{C}'$  is a subset of  $\pi_C(C)$ . The definable family of cells  $\mathcal{C} = \{\pi_C^{-1}(C') : C' \in \mathcal{C}'\}$  is then downward directed, complete, and finer than  $\mathcal{S}$ .

Hence onwards we assume that, for every cell  $C \subseteq M^n$  with  $\dim C < n$ , there exists some  $S \in \mathcal{S}$  such that  $S \cap C = \emptyset$ . By Fact 5.2.4 and o-minimal cell decomposition it follows that, for every definable set  $X \subseteq M^n$ ,

$$\text{if } \dim X < n \text{ then there exists } S \in \mathcal{S} \text{ such that } S \cap X = \emptyset. \quad (\dagger)$$

In particular  $\dim S = n$  for any  $S \in \mathcal{S}$ .

Consider the definable downward directed family  $\pi(\mathcal{S}) = \{\pi(S) : S \in \mathcal{S}\}$ . By induction hypothesis there exists a definable downward directed family of cells  $\mathcal{B}$  in  $M^{n-1}$  that is complete and finer than  $\pi(\mathcal{S})$ . Let  $\mathcal{F} = \{S \cap (B \times M) : S \in \mathcal{S}, B \in \mathcal{B}\}$ . We show that this definable family is downward directed.

Let  $S, S' \in \mathcal{S}$  and  $B, B' \in \mathcal{B}$ . Let  $S'' \in \mathcal{S}$  be such that  $S'' \subseteq S \cap S'$ . Since  $\mathcal{B}$  is finer than  $\pi(\mathcal{S})$  and downward directed there exists  $B'' \in \mathcal{B}$  with  $B'' \subseteq \pi(S'') \cap B \cap B'$ . Then  $\emptyset \neq S'' \cap (B'' \times M) \subseteq S \cap (B \times M) \cap S' \cap (B' \times M)$ .

Clearly  $\mathcal{F}$  is finer than  $\mathcal{S}$ . We proceed by proving the lemma for  $\mathcal{F}$  in place of  $\mathcal{S}$ .

Let  $\mathcal{F} = \{\varphi(u, M^n) : u \in \Omega \subseteq M^m\}$ . Let  $\mathcal{D}$  be a cell decomposition of  $\varphi(M^m, M^n)$ . Then, by uniform cell decomposition, for every  $u \in \Omega$  the family of fibers  $\{D_u : D \in \mathcal{D}, D_u \neq \emptyset\}$  is a cell decomposition of  $\varphi(u, M^n)$  and the family  $\{\pi(D_u) : D \in \mathcal{D}, D_u \neq \emptyset\}$  is a cell decomposition of  $\pi(\varphi(u, M^n))$ . For each  $u \in \Omega$  let  $\mathcal{D}_u = \{D_u : D \in \mathcal{D}\}$ .

For the rest of the proof we assume that, for every  $F \in \mathcal{F}$ , there is a unique  $u \in \Omega$  with  $F = \varphi(u, M^n)$ . Then for clarity we identify  $F$  with  $u$  by writing  $D_F$  in place of  $D_u$  and  $\mathcal{D}_F$  in place of  $\mathcal{D}_u$ , onwards omitting the subscript  $u$  entirely. This assumption is valid if  $\mathcal{M}$  has

elimination of imaginaries. Nevertheless it is adopted entirely for clarity and the proof can be written in terms of  $u \in \Omega$  instead of  $F = \varphi(u, M^n)$ .

We now wish to show that there exists a definable family of open cells  $\mathcal{C} = \{C_F \in \mathcal{D}_F : F \in \mathcal{F}\}$  such that, for every  $F \in \mathcal{F}$ :

- (i)  $C_F \cap F' \neq \emptyset$  for every  $F' \in \mathcal{F}$ .
- (ii) For every  $C \in \mathcal{D}_F$ , if  $C \cap F' \neq \emptyset$  for every  $F' \in \mathcal{F}$ , then  $C = C_F$  or  $\pi(C) = \pi(C_F)$  and  $C_F < C$ , i.e. if  $C_F = (f, g)$  and  $C = (f', g')$  then  $g \leq f'$ .

Note that, for every  $F \in \mathcal{F}$ , there can be at most one  $D \in \mathcal{D}$  such that  $D_F = C_F$  (i.e. satisfying (i) and (ii) above).

Observe that, for every  $D \in \mathcal{D}$ , the family of  $F \in \mathcal{F}$  such that  $D_F = C_F$  is definable. Consider these induced subfamilies of  $\mathcal{F}$ , as  $D$  ranges in  $\mathcal{D}$ . If  $\mathcal{C}$  as described above exists, then these subfamilies cover  $\mathcal{F}$ . Additionally, by downward directedness, one of them must be finer than  $\mathcal{F}$ . Hence, if it exists,  $\mathcal{C}$  is definable, and in particular, by passing to a subfamily of  $\mathcal{F}$  if necessary, we may in fact assume that  $\mathcal{C}$  is a family of fibers over a single cell in  $\mathcal{D}$ . We show that  $\mathcal{C}$  exists.

For any  $F \in \mathcal{F}$  let  $\mathcal{D}'_F := \{C \in \mathcal{D}_F : C \cap F' \neq \emptyset \forall F' \in \mathcal{F}\}$ . By Fact 5.2.4 this family is nonempty. By (†) every cell in  $\mathcal{D}'_F$  must be open. Now note that, since  $\mathcal{B}$  is complete and, by construction,  $\pi(\mathcal{F})$  is finer than  $\mathcal{B}$ ,  $\pi(\mathcal{F})$  is complete. By Fact 5.2.5 it follows that, for every  $F \in \mathcal{F}$ , there exists a unique set in  $\pi(\mathcal{D}_F)$ , say  $B_F$ , such that  $\pi(F') \subseteq B_F$  for some  $F' \in \mathcal{F}$ . Hence, for any  $C, C' \in \mathcal{D}'_F$ , it holds that  $\pi(C) = \pi(C') = B_F$ . Let  $C_F = (f, g)$  be the cell in  $\mathcal{D}'_F$  satisfying that, for any other cell  $(f', g') \in \mathcal{D}'_F$ ,  $g \leq f'$ .

We have defined  $\mathcal{C} = \{C_F : F \in \mathcal{F}\}$ . This family is clearly finer than  $\mathcal{F}$ . We prove that it is downward directed and complete.

**Claim 5.2.7.** *For every pair  $F, F' \in \mathcal{F}$ ,  $C_F \cap C_{F'} \neq \emptyset$ .*

Let  $C_F = (f, g)$  and  $C_{F'} = (f', g')$ , and suppose towards a contradiction that  $C_F \cap C_{F'} = \emptyset$ . Recall that  $\pi(\mathcal{F})$  is downward directed and complete, and notation  $B_F$  for the projection of any (every) cell in  $\mathcal{D}'_F$ . By (i) there exists some  $B \in \pi(\mathcal{F})$  such that  $B \subseteq B_F \cap B_{F'}$ . Without loss of generality suppose that  $g'|_B \leq f|_B$ .

Let  $\mathcal{D}_F \setminus \mathcal{D}'_F = \{C(1), \dots, C(l)\}$ . By definition of  $\mathcal{D}'_F$  for every  $1 \leq i \leq l$  there exists some  $F(i) \in \mathcal{F}$  such that  $C(i) \cap F(i) = \emptyset$ . Let  $F'' \in \mathcal{F}$  be such that  $F'' \subseteq F \cap (\bigcap_{i=1}^l F(i)) \cap (B \times M)$ . Then, by (ii), we have that  $F'' \subseteq (f|_B, +\infty)$ . But then  $F'' \cap C_{F'} = \emptyset$ , contradicting (i).

Let  $C_F = (f, g)$  and  $C_{F'} = (f', g')$ , and suppose towards a contradiction that  $C_F \cap C_{F'} = \emptyset$ . By (i) and because  $\mathcal{B}$  is complete there exists  $B \in \mathcal{B}$  such that  $B \subseteq \pi(C_F) \cap \pi(C_{F'})$ . Without loss of generality suppose that  $g|_B \leq f'|_B$ . By definition of  $\mathcal{D}'_{F'}$  and because  $\mathcal{F}$  is downward directed there exists some  $F''' \in \mathcal{F}$  such that  $F''' \subseteq \bigcup \mathcal{D}'_{F'}$ . Let  $F'''' \in \mathcal{F}$  be such that  $F'''' \subseteq F''' \cap B \times M$ . Then, by (ii),  $F'''' \subseteq (f'|_B, +\infty)$ , and in particular  $F'''' \cap C_F = \emptyset$ , contradiction.

**Claim 5.2.8.** *The family  $\mathcal{C}_{\mathcal{F}}$  is downward directed.*

Let  $C, C' \in \mathcal{C}_{\mathcal{F}}$ . Recall that  $\mathcal{F}$  is finer than  $\mathcal{S}$ . By (†) let  $F \in \mathcal{F}$  be such that  $F \cap \partial C = \emptyset$  and  $F \cap \partial C' = \emptyset$ . By Claim 5.2.7  $C_F \cap C \neq \emptyset$ . Since every cell is definably connected and every cell in  $\mathcal{C}_{\mathcal{F}}$  is open, it follows that  $C_F \subseteq C$ . By the same argument  $C_F \subseteq C'$ . Hence  $C_F \subseteq C \cap C'$ .

Finally we show that  $\mathcal{C}$  is complete. Suppose otherwise, in which case there exists a definable set  $X \subseteq M^n$  such that, for every  $C \in \mathcal{C}$ ,  $C \cap X \neq \emptyset$  and  $C \setminus X \neq \emptyset$ . By (†),  $\dim X = n$ , and we may find  $F \in \mathcal{F}$  such that  $F \cap \text{bd}(X) = \emptyset$ , having in particular that  $C_F \cap \text{bd}(X) = \emptyset$ . It follows that  $C_F$  is not definably connected, contradicting that  $C_F$  is a cell.  $\square$

Notice that, in the base case of the proof of Lemma 5.2.6, if  $\mathcal{S}$  is definable then the “ $a$ ” is definable over the same parameters as  $\mathcal{S}$ . It follows that  $\mathcal{C}$  is definable over the same parameters as  $\mathcal{S}$ . We might ask if this holds in higher dimensions. If  $\mathcal{M}$  has definable choice functions then it does, by the usual argument involving passing to an elementary substructure. The paper in preparation by the author [1] on which this chapter is based includes an appendix by Will Johnson in which he proves that, in general within o-minimality, the answer to that question is no.

We now present a second lemma connecting types and downward directed families.

**Lemma 5.2.9.** *Let  $p$  be an  $n$ -type and  $\mathcal{S}$  be a uniform family of subsets of  $M^n$ . There exists a uniform downward directed family of sets  $\mathcal{F} \subseteq p$  that is complete for  $\mathcal{S}$ . If  $p$  is definable then  $\mathcal{F}$  can be chosen definable over the same parameters as  $p$ .*

*Proof.* Since otherwise the result is immediate we may assume that  $p$  is not realised in  $\mathcal{M}$ . We proceed by induction on  $n$ . For simplicity we prove the case where  $p$  is definable. The case where  $p$  is not definable follows by the same arguments. The fact that  $\mathcal{F}$  can be chosen definable over the same parameters as  $p$  follows by keeping track of parameters.

We start by reducing the proof to the case where  $\mathcal{S}$  is a definable family of cells and  $\mathcal{S} \subseteq p$ .

By expanding  $\mathcal{S}$  if necessary let us assume that it is a family of the form  $\mathcal{S} = \{\varphi(u, M^n) : u \in M^m\}$ . Let  $\mathcal{D}$  be a cell decomposition of  $M^m \times M^n$  compatible with  $\varphi(M^m, M^n)$ . Then, for every  $u \in M^m$ , the family of fibers  $\{D_u : D \in \mathcal{D}\}$  is a partition of  $M^n$  compatible with  $\varphi(u, M^n)$ . This means that there exists some  $D \in \mathcal{D}$  such that  $D_u \in p$ , and moreover either  $D_u \subseteq \varphi(u, M^n)$  or  $D_u \cap \varphi(u, M^n) = \emptyset$ .

Consider the union  $\mathcal{C}$  of the families  $\{D_u : u \in M^m, D_u \in p\}$  for  $D \in \mathcal{D}$ . Then  $\mathcal{C}$  is a definable family of cells with  $\mathcal{C} \subseteq p$  and, for every  $S \in \mathcal{S}$ , there is some  $C \in \mathcal{C}$  such that either  $C \subseteq S$  or  $C \cap S = \emptyset$ . Clearly it suffices to prove the lemma for  $\mathcal{C}$  in place of  $\mathcal{S}$ .

Hence onwards we assume that  $\mathcal{S}$  is a definable family of cells and  $\mathcal{S} \subseteq p$ . This means in particular that we must prove the existence of a downward directed definable family  $\mathcal{F} \subseteq p$  that is finer than  $\mathcal{S}$ .

Suppose that  $n = 1$ . Then the result follows from the fact that every non-realised 1-type has a uniform basis of intervals. In particular when the type is definable this basis is of the form  $\{(a, t), t > a\}$  for some  $a \in M \cup \{-\infty\}$ , or  $\{(t, a) : t < a\}$  for some  $a \in M \cup \{+\infty\}$ . These are complete definable families of nested sets.

Suppose that  $n > 1$ . Suppose there exists  $S^* \in \mathcal{S}$  that is the graph of some definable continuous function  $\pi(S^*) \rightarrow M$ . Note that the type  $\pi(p) = \{\pi(X) : X \in p\}$  is definable. Note that the definable family  $\mathcal{S} \cap \mathcal{S} = \{S \cap S' : S, S' \in \mathcal{S}\}$  is a subset of  $p$ . We apply the induction hypothesis to the type  $\pi(p)$  and family  $\pi(\mathcal{S} \cap \mathcal{S})$  and reach a downward directed definable family  $\mathcal{G} \subseteq \pi(p)$  that is finer than  $\pi(\mathcal{S} \cap \mathcal{S})$ . Let  $\mathcal{F} = \{S \cap (G \times M) : S \in \mathcal{S}, G \in \mathcal{G}\}$ .

Then  $\mathcal{F}$  is clearly definable and finer than  $\mathcal{S}$ . We show that it is a subset of  $p$  and downward directed. Firstly, since  $\mathcal{G} \subseteq \pi(p)$ , for every  $G \in \mathcal{G}$  it holds that  $G \times M \in p$ , and thus  $S \cap (G \times M) \in p$  for every  $S \in \mathcal{S}$ . Secondly, since  $\mathcal{G}$  is downward directed and finer than  $\pi(\mathcal{S} \cap \mathcal{S})$ , for every  $S, S' \in \mathcal{S}$  there exists  $G \in \mathcal{G}$  such that  $G \subseteq \pi(S^* \cap S) \cap \pi(S^* \cap S')$ . Hence in particular  $S^* \cap (G \times M) \subseteq S \cap S'$ . It follows that, for any  $G', G'' \in \mathcal{G}$ , if  $G''' \subseteq G \cap G' \cap G''$ , then  $S^* \cap (G''' \times M) \subseteq S \cap (G' \times M) \cap S' \cap (G'' \times M)$ .

We now prove the case where every cell  $S \in \mathcal{S}$  is of the form  $(f_S, g_S)$  for definable continuous functions  $f_S, g_S : \pi(S) \rightarrow M_{\pm\infty}$ . Consider the definable families of sets  $\mathcal{S}_0 = \{(-\infty, g_S) : S \in \mathcal{S}\}$  and  $\mathcal{S}_1 = \{(f_S, +\infty) : S \in \mathcal{S}\}$ . We observe that it suffices to find definable downward directed families  $\mathcal{F}_0$  and  $\mathcal{F}_1$  contained in  $p$  such that  $\mathcal{F}_0$  is complete for  $\mathcal{S}_0$  and  $\mathcal{F}_1$  is complete for  $\mathcal{S}_1$ . In that case let  $\mathcal{F} = \{F \cap F' : F \in \mathcal{F}_0, F' \in \mathcal{F}_1\}$ . Then  $\mathcal{F}$  is clearly contained in  $p$ , in particular it does not contain the empty set, and is downward directed. Moreover, for any  $S = (f, g) \in \mathcal{S}$ , if  $F_0 \in \mathcal{F}_0$  is such that  $F_0 \subseteq (-\infty, g)$  and  $F_1 \in \mathcal{F}_1$  is such that  $F_1 \subseteq (f, +\infty)$ , then  $F_0 \cap F_1 \subseteq S$ , so  $\mathcal{F}$  is finer than  $\mathcal{S}$ . We prove the existence of  $\mathcal{F}_0$  and  $\mathcal{F}_1$ .

For any two  $S, S' \in \mathcal{S}$  let  $X(S, S') = \{x \in \pi(S) \cap \pi(S') : g_S(x) \leq g_{S'}(x)\}$ . These sets are definable uniformly over  $S, S' \in \mathcal{S}$  (formally over  $\Omega^2$ , if  $\Omega$  is the index set of  $\mathcal{S}$ ). Let  $\mathcal{X} = \{X(S, S') : S, S' \in \mathcal{S}\}$ . By induction hypothesis there exists a definable downward directed family  $\mathcal{B} \subseteq \pi(p)$  that is complete for  $\mathcal{X}$ . Let  $\mathcal{F}_0 = \{(B \times M) \cap (-\infty, g_S) : B \in \mathcal{B}, S \in \mathcal{S}\}$ . We claim that this family is downward directed, contained in  $p$  and finer than  $\mathcal{S}_0$ . The last property is obvious. Moreover, for every  $B \in \mathcal{B}$ , since  $B \in \pi(p)$  then  $B \times M \in p$  and so, for every  $S \in \mathcal{S}$ ,  $(B \times M) \cap (-\infty, g_S) \in p$ . It remains to prove that  $\mathcal{F}_0$  is downward directed.

Let us fix  $B', B'' \in \mathcal{B}$  and  $S, S' \in \mathcal{S}$ . Note that  $\pi(S) \cap \pi(S')$  is covered by  $X(S, S')$  and  $X(S', S)$ . Since  $\mathcal{B}$  is a subset of  $\pi(p)$  and complete for  $\mathcal{X}$  it follows that there exists  $B''' \in \mathcal{B}$  satisfying either  $B''' \subseteq X(S, S')$  or  $B''' \subseteq X(S', S)$  (fact 5.2.5). Suppose without loss of generality that it is the former. Then  $(B''' \times M) \cap (-\infty, g_S) \subseteq (-\infty, g_{S'})$ . Let  $B \in \mathcal{B}$  be such that  $B \subseteq B' \cap B'' \cap B'''$ , then  $(B \times M) \cap (-\infty, g_S) \subseteq (B' \times M) \cap (-\infty, g_S) \cap (B'' \times M) \cap (-\infty, g_{S'})$ . We have shown that  $\mathcal{S}_0$  is downward directed.

The construction of  $\mathcal{S}_1$  is analogous. This completes the proof of the lemma.  $\square$

**Remark 5.2.10.** Let  $p$  be a definable type and  $\mathcal{S} \subseteq p$  be an  $A$ -definable family of sets. Is there always an  $A$ -definable downward directed family  $\mathcal{F} \subseteq p$  finer than  $\mathcal{S}$ ?

The negative answer is given by the following counterexample. Consider  $\mathcal{S} = \{M \setminus \{t\} : t \in M\}$ , a family which is  $\emptyset$ -definable, and let  $p$  be a non-realised type with basis  $\{(s_0, t) : t > s_0\}$  for some fixed  $s_0$  with  $s_0 \notin \text{dcl}(\emptyset)$ . Suppose there exists  $\mathcal{F}$  as above and  $\emptyset$ -definable. Consider  $B = \{s \in M : \forall F \in \mathcal{F} \exists t > s (s, t) \subseteq F\}$ . Clearly  $B$  is  $\emptyset$ -definable. Since  $\mathcal{F} \subseteq p$  the point  $s_0$  is in  $B$ . If  $B$  is finite then  $s_0 \in \text{dcl}(\emptyset)$ , contradiction. Suppose that  $B$  contains an interval  $I$ . By definition of  $B$  and o-minimality this means that every set  $F \in \mathcal{F}$  must satisfy that  $F \cap I$  is cofinite in  $I$ . By uniform finiteness there is an  $m$  such that  $|I \setminus (F \cap I)| < m$  for every  $F \in \mathcal{F}$ . This however contradicts that  $\mathcal{F}$  is downward directed and finer than  $\mathcal{S}$ .

On the other hand in the next section we prove Proposition 5.3.18, which implies that  $\mathcal{S}$  as above always extends to an  $A$ -definable type. By virtue of Lemma 5.2.9 there exists an  $A$ -definable downward directed family finer than  $\mathcal{S}$ .

We may now prove the main result of this section.

**Theorem 5.2.11.** *Let  $\mathcal{S} \subseteq \mathcal{P}(M^n)$  be a uniform family of sets. The following are equivalent (with and without the definability condition in parentheses).*

- (1)  $\mathcal{S}$  extends to a (definable)  $n$ -type.
- (2) There exists a uniform (definable) downward directed family finer than  $\mathcal{S}$ .
- (3)  $\mathcal{S}$  extends to a (definable)  $n$ -type with a uniform basis of cells.

*Proof.* (1)  $\Rightarrow$  (2) is given by Lemma 5.2.9, (2)  $\Rightarrow$  (3) by Lemma 5.2.6, (3)  $\Rightarrow$  (1) is trivial.  $\square$

**Remark 5.2.12.** Theorem 5.2.11 highlights an interesting property of types with uniform bases in o-minimal structures. Consider the “uniform topology” on  $S_n(M)$  where basic open sets  $A_{\mathcal{S}}$  are indexed by uniform families  $\mathcal{S} \subseteq \text{Def}(M^n)$ , where  $A_{\mathcal{S}} = \{p \in S_n(M) : \mathcal{S} \subseteq p\}$ . This topology is clearly finer than the usual Stone topology among types. Note that, in this topology, the types with a uniform basis are precisely the isolated types. Moreover for every  $A \subseteq M$  the set of all  $A$ -definable types is closed. In the context of this topology

Theorem 5.2.11 states that isolated types are dense in  $S_n(M)$  and isolated definable types are dense in the closed subspace of all definable types.

Recall that the appendix by Will Johnson in [1] shows that a parameter version of Lemma 5.2.6 is not possible. That is, it is not true in general that every  $A$ -definable downward directed family of sets extends to an  $A$ -definable type with a uniform basis. We end this section however by proving the next best thing, that every  $A$ -definable downward directed family extends to an  $A$ -definable type.

The following method to construct types will be useful in Proposition 5.2.15 and further on.

**Definition 5.2.13.** *Let  $p$  be a  $n$ -type and  $f : M^n \rightarrow M$  be a definable function with  $\text{dom}(f) \in p$ . We define three  $(n + 1)$ -types.*

*Let  $f|_p$  be the type of sets  $S$  such that  $\pi(S \cap \text{graph}(f)) \in p$ .*

*Let  $f^+|_p$  be the type of sets  $S$  such that  $\{x \in \text{dom}(f) : \{x\} \times (f(x), s) \subseteq S \text{ for some } s > f(x)\} \in p$ .*

*Let  $f^-|_p$  be the type of sets  $S$  such that  $\{x \in \text{dom}(f) : \{x\} \times (s, f(x)) \subseteq S \text{ for some } s < f(x)\} \in p$ .*

*The definitions of  $f^+|_p$  and  $f^-|_p$  hold too if  $f$  is, respectively, the constant function  $-\infty$  and  $+\infty$ .*

The next lemma without proof follows by routine application of o-minimality and knowledge of the Fiber Lemma for o-minimal dimension (Lemma 2.1.3).

**Lemma 5.2.14.** *Given an  $n$ -type  $p$  and definable function  $f : M^n \rightarrow M$  with  $\text{dom}(f) \in p$  the types  $f|_p$ ,  $f^+|_p$  and  $f^-|_p$  are well defined. If  $p$  is  $A$ -definable and  $f$   $B$ -definable the they are  $AB$ -definable. If  $p$   $d$ -dimensional then  $f|_p$  is  $d$ -dimensional and  $f^+|_p$  and  $f^-|_p$  are  $(d + 1)$ -dimensional.*

**Proposition 5.2.15.** *Let  $A \subseteq M$  and let  $\mathcal{S} \subseteq \text{Def}(M^n)$  be an  $A$ -definable downward directed family of sets. Then  $\mathcal{S}$  extends to an  $A$ -definable type.*

*Proof.* We proceed by induction on  $n$ . If  $n = 1$  then this is shown by the base case in the proof of Lemma 5.2.6. Suppose that  $n > 1$ .

By induction hypothesis let  $p$  be an  $A$ -definable type expanding  $\pi(\mathcal{S})$ . For every  $S \in \mathcal{S}$  we define  $f_S : \pi(S) \rightarrow M_{\pm\infty}$  by  $f_S(u) = \sup S_u$ , where  $S_u$  is the corresponding fiber  $\{t : \langle u, t \rangle \in S\}$ . Let  $\mathcal{F} = \{f_S : S \in \mathcal{S}\}$ . This family is clearly  $A$ -definable. Let  $\preceq$  be the  $A$ -definable total preorder induced by  $p$  described in Section 5.1.1. We consider a number of cases.

**Case 1:**  $\mathcal{F}$  has a minimum with respect to  $\preceq$ . Let  $\mathcal{S}_{\min} \subseteq \mathcal{S}$  be the family of  $S$  such that  $f_S$  is a minimum. Let  $\mathcal{F}_{\min} = \{f_S : S \in \mathcal{S}_{\min}\}$ . We prove that, for any given  $S \in \mathcal{S}_{\min}$ ,  $\mathcal{S}$  extends to either  $f_S|_p$  or  $f_S^-|_p$ , and show that these types are  $A$ -definable. We make use of two observations.

**Claim 5.2.16.** *For every  $g, h \in \mathcal{F}_{\min}$ ,  $g|_p = h|_p$  and  $g^-|_p = h^-|_p$ . We denote these types  $\mathcal{F}_{\min}|_p$  and  $\mathcal{F}_{\min}^-|_p$  respectively.*

Since the family  $\mathcal{F}_{\min}$  is  $A$ -definable, it follows from Claim 5.2.16 and definition 5.2.13 that  $\mathcal{F}_{\min}|_p$  and  $\mathcal{F}_{\min}^-|_p$  are  $A$ -definable. We prove the claim.

Let  $g, h \in \mathcal{F}_{\min}$ . Observe that, because  $g \preceq h$  and  $h \preceq g$ , we have that  $\{x : g(x) = h(x)\} \in p$ . Let  $Z \in \text{Def}(M^n)$  be such that  $Z \in g|_p$ . Then  $\{x : \langle x, h(x) \rangle \in Z\} \supseteq \{x : g(x) = h(x), \langle x, g(x) \rangle \in Z\} \in p$ . So  $Z \in h|_p$ . Analogously one proves that  $g^-|_p = h^-|_p$ .

**Fact 5.2.17.** *For every  $S \in \mathcal{S}$ , exactly one of the sets  $\{x : f_S(x) \in S_x\}$  and  $\{x : f_S(x) \notin S_x\}$  must belong in  $p$ . If it is the former then  $S \in f_S|_p$ . If it is the latter then, by definition of  $f_S$  and  $o$ -minimality, we have that  $S \in f_S^-|_p$ .*

We assume that  $\mathcal{S}$  does not extend to  $\mathcal{F}_{\min}|_p$  and prove that it extends to  $\mathcal{F}_{\min}^-|_p$ . Let  $S^* \in \mathcal{S}$  be such that  $S^* \notin \mathcal{F}_{\min}|_p$ . Let  $S' \in \mathcal{S}_{\min}$ . For any  $S \in \mathcal{S}$ , by downward directedness there exists  $S'' \in \mathcal{S}$  with  $S'' \subseteq S \cap S' \cap S^*$ . Since  $S'' \subseteq S'$  we have that  $S'' \in \mathcal{S}_{\min}$ . Since  $S'' \subseteq S^*$ ,  $S'' \notin \mathcal{F}_{\min}|_p = f_{S''}|_p$ . By Fact 5.2.17 it follows that  $S'' \in f_{S''}^-|_p$ . We conclude that  $S \in f_{S''}^-|_p = \mathcal{F}_{\min}^-|_p$ .

**Case 2:**  $\mathcal{F}$  does not have a minimum with respect to  $\preceq$ . Consider the  $n$ -type  $q$  given by all sets  $F \in \text{Def}(M^n)$  such that  $F \in f_S|_p$  for every  $S$  in some subfamily  $\mathcal{S}' \subseteq \mathcal{S}$  that is unbounded from below with respect to  $\preceq$ . We prove the consistency of this type. For this it suffices to show that, for every cell  $C \in \text{Def}(M^n)$ , if  $C \in q$ , then there is some  $S' \in \mathcal{S}$



such that  $C \in f_S|_p$  for every  $S \in \mathcal{S}$  with  $S \preceq S'$ . However this follows from the fact that, if  $C \in q$ , it must be an open cell of the form  $C = (f, g)$  with  $f \prec \mathcal{F}$  and  $f_S \preceq g$  for some  $S \in \mathcal{S}$ . Finally note that, since  $\mathcal{F}$  and  $\preceq$  are  $A$ -definable, by Definition 5.2.13 this type is  $A$ -definable.  $\square$

**Remark 5.2.18.** Lemma 5.2.9 and Proposition 5.2.15 show that, in the definable statement of Theorem 5.2.11, the type in (1) and the downward directed family in (2) can be required to be definable over the same parameters.

On the other hand Will Johnson proved ([1], Appendix B) that the type with a uniform basis in (3) might not exist definable over these same parameters.

In the context of the topology described in Remark 5.2.12, it is not true in general that, for any  $A \subseteq M$ , the family of isolated  $A$ -definable types is dense in the closed space of all  $A$ -definable types.

### 5.3 Transversals

Throughout this section we fix a monster model  $\mathcal{U} = (U, <, \dots)$  of  $Th(\mathcal{M})$ . For a definable set  $X$  let  $X(\mathcal{U})$  denote the interpretation on  $X$  in  $\mathcal{U}$ . Our main result is Theorem 5.3.9, stating that if a definable family has the  $(m, n)$ -property for some large enough  $n$  and  $m \geq n$ , then it partitions into finitely many subfamilies, each of which extends to a definable type.

**Definition 5.3.1.** *Let  $\mathcal{S} \subseteq Def(M^n)$ . A transversal  $F$  of  $\{S(\mathcal{U}) : S \in \mathcal{S}\}$  is called tame if it belongs in some tame extension of  $\mathcal{M}$ . We abuse terminology by calling  $F$  a tame transversal of  $\mathcal{S}$ .*

Let  $\mathcal{S}$  be a uniform family. We are interested in the property that  $\mathcal{S}$  admits a tame transversal that is finite. We give an equivalent definition of this property.

**Definition 5.3.2.** *Let  $\mathcal{S} = \{\varphi(u, M^n) : u \in \Omega \subseteq M^m\}$  be a uniform family and  $A \subseteq M$ . We say that  $\mathcal{S}$  has a finite tame transversal (FTT) over  $A$  if there are sets  $\Omega_1, \dots, \Omega_k \subseteq \Omega$  with  $\cup_i \Omega_i = \Omega$  such that, for every  $1 \leq i \leq k$ , the family  $\mathcal{S}_i = \{\varphi(u, M^n) : u \in \Omega_i\}$  extends to an  $A$ -definable type.*

*Without mention to a specific  $A$  we mean  $A = M$ .*

By the Marker-Steinhorn Theorem (Theorem 2.1.8) every definable type is realized in some tame elementary extension of  $\mathcal{M}$ . So both definitions of having FTT are equivalent.

We precede Theorem 5.3.9 with two lemmas. First however we introduce some notation on cuts.

Given a definable totally preordered set  $(X, \preceq)$ , a *cut*  $(P, Q)$  of  $(X, \preceq)$  is a partition of  $X$  into two sets  $P$  and  $Q$  such that  $x \prec y$  for every  $x \in P$  and  $y \in Q$ . Such a cut  $(P, Q)$  is definable if  $P$  (equivalently  $Q$ ) is definable. All the cuts we consider satisfy that either  $P$  does not have a maximum or  $Q$  does not have a minimum with respect to  $\preceq$ . That is, for every  $x \in P$  and  $y \in Q$  the interval  $(x, y)_{\preceq}$  is nonempty. Consequently we may always associate to a cut  $(P, Q)$  the partial type  $\{(x, y)_{\preceq} : x \in P, y \in Q\}$ , and by means of this association we often refer to cuts as types. In particular we say that  $z \in X(\mathcal{U})$  realizes  $(P, Q)$  if  $(x, y)_{\preceq} \in \text{tp}(z/M)$  for every  $x \in P$  and  $y \in Q$ .

Without context, a cut is a cut in  $(M, \leq)$ . In this setting non-definable and definable cuts are denoted, respectively, irrational and rational cuts in [41], and simply cuts and noncuts in other sources such as [31] and [12]. By o-minimality, any non-isolated 1-type over  $M$  is uniquely characterised by the unique cut it realizes in  $(M, \leq)$ .

It is worth noting that, in Lemma 5.3.3 below, the existence of the uniform bound  $l$  is redundant, since it follows from o-minimality.

**Lemma 5.3.3.** *Let  $(X, \preceq)$ ,  $X \subseteq M^k$ , be a definable totally preordered set, and  $l < \omega$ . Let  $\mathcal{S}$  be a definable family of nonempty subsets of  $X$ , all of which are finite union of at most  $l$  points and intervals with respect to  $\preceq$  with endpoints in  $X \cup \{-\infty, +\infty\}$  (where  $-\infty$  and  $+\infty$  have the natural interpretation with respect to  $\preceq$ ). Then exactly one of the following holds:*

- (1)  $\mathcal{S}$  has a finite tame transversal.
- (2) There exists  $\mathcal{S}' \subseteq \mathcal{S}$  an infinite subfamily (not necessarily definable) of pairwise disjoint sets.

*Proof.* Clearly (1) and (2) are mutually exclusive. We assume the negation of (1) and prove (2). Throughout the proof let the term interval refer to an interval in  $(X, \preceq)$  with endpoints in  $X \cup \{-\infty, +\infty\}$ .

Hence let us assume that  $\mathcal{S} = \{\varphi(u, M^k) : u \in \Omega\}$  does not admit a finite tame transversal. Then by model theoretic compactness there exists  $c \in \Omega(\mathcal{U})$ ,  $\mathbb{S} = \varphi(c, U^k)$ , with the property that, for every definable  $\Omega' \subseteq \Omega$ , if  $\{\varphi(u, M^k) : u \in \Omega'\}$  extends to a definable type, then  $c \notin \Omega'(\mathcal{U})$ . In particular this holds for  $\Omega' = \Omega_x = \{u \in \Omega : \mathcal{M} \models \varphi(u, x)\}$  for every  $x \in X$ , and so  $\mathbb{S} \cap X = \emptyset$ . Moreover note that, if  $\sim$  denotes the equivalence relation induced by  $\preceq$  on  $X$ , then by definition every  $S \in \mathcal{S}$  is compatible with the equivalence classes of  $\sim$ , i.e. for every  $x, y \in X$  if  $x \sim y$  and  $x \in S$  then  $y \in S$  too. This must also be satisfied by  $\mathbb{S}$  with respect to the interpretation of  $\preceq$  on  $X(\mathcal{U})$  (which we also denote  $\preceq$ ). So each equivalence class in  $X(\mathcal{U})$  of points in  $X$  must be disjoint from  $\mathbb{S}$ . Hence each point  $x \in \mathbb{S}$  induces a cut  $(P, Q)$  in  $X$  by  $P = \{y \in X : y \prec x\}$  and  $Q = X \setminus P$ .

Let  $(P, Q)$  and  $(P', Q')$  be different cuts in  $M$ , with, say,  $P \subsetneq P'$ , and let  $x_{(P, P')} \in P' \setminus P$ . Since  $x_{(P, P')} \in X$  we have that  $x_{(P, P')} \notin \mathbb{S}$  and so, if  $x$  and  $y$  are points in  $\mathbb{S}$  that realize  $(P, Q)$  and  $(P', Q')$  respectively, then these points do not belong in the same subinterval of  $\mathbb{S}$ . Since every set in  $\mathbb{S}$  is union of at most  $l$  points and intervals, there are at most  $l$  many distinct cuts  $(P_i, Q_i)$ ,  $0 \leq i \leq m$ , such that every point in  $\mathbb{S}$  realizes one of these cuts.

For simplicity we assume that all the cuts  $(P_i, Q_i)$  are such that  $P_i \neq \emptyset$  and  $Q_i \neq \emptyset$ . The proof readily adapts to the other case. Let the indexing be such that  $(P_i, Q_i)$ , for  $0 \leq i \leq n$ , are the cuts that are non-definable, and  $(P_i, Q_i)$ , for  $n < i \leq m$ , are definable.

For every  $0 \leq i \leq m$  let us fix  $a_i \in P_i$  and  $b_i \in Q_i$  such that  $\{a_i, b_i\} \subseteq P_j$  or  $\{a_i, b_i\} \subseteq Q_j$  for every  $j \neq i$ . We show that there exists  $S \in \mathcal{S}$  with  $S \subseteq \bigcup_i (a_i, b_i)_{\preceq}$  such that, for every  $0 \leq i \leq m$ , there are  $a'_i \in P_i$ ,  $b'_i \in Q_i$ ,  $a_i \preceq a'_i \prec b'_i \preceq b_i$ , with  $(a'_i, b'_i)_{\preceq} \cap S = \emptyset$  (in particular  $S \cap \mathbb{S} = \emptyset$ ). We may then apply the result again with parameters  $a'_i, b'_i$  in place of  $a_i$  and  $b_i$  and reach a second set  $S' \in \mathcal{S}$  that will be disjoint from  $S$ . Repeating this process yields a countably infinite pairwise disjoint subfamily of  $\mathcal{S}$  as desired.

Let  $(P, Q)$  be a cut. Whenever a definable set  $S$  satisfies that  $(a, b)_{\preceq} \cap S = \emptyset$  for some  $a \in P$ ,  $b \in Q$ , we say that  $S$  is disjoint from  $(P, Q)$ . Hence we must find  $S \in \mathcal{S}$  contained in  $\bigcup_i (a_i, b_i)_S$  that is disjoint from every cut  $(P_i, Q_i)$ ,  $0 \leq i \leq m$ .

We focus first on the definable cuts.

**Claim 5.3.4.** *Let  $(P, Q)$  be a definable cut in  $X$ , then  $\mathbb{S}$  is disjoint from the cut  $(P(\mathcal{U}), Q(\mathcal{U}))$ .*

We show that there is not  $a \in P(\mathcal{U})$  and  $b \in Q(\mathcal{U})$  with  $(a, b)_{\preccurlyeq(\mathcal{U})} \subseteq \mathbb{S}$ . Since  $\mathbb{S}$  is union of finitely many points and intervals with respect to  $\preccurlyeq$  it follows that there exists  $a \in P(\mathcal{U})$  and  $b \in Q(\mathcal{U})$  with  $(a, b)_{\preccurlyeq(\mathcal{U})} \cap \mathbb{S} = \emptyset$ .

Suppose that there are  $a \in P(\mathcal{U})$  and  $b \in Q(\mathcal{U})$  with  $(a, b)_{\preccurlyeq(\mathcal{U})} \subseteq \mathbb{S}$ . Let  $\Sigma \subseteq \Omega$  be denote the set of index elements  $u$  such that  $(a, b)_{\preccurlyeq} \subseteq \varphi(u, M^k)$  for some  $a \in P, b \in Q$ . Since the cut is definable then so is  $\Sigma$ . Moreover  $c$ , where  $\mathbb{S} = \varphi(c, U^k)$ , belongs in  $\Sigma(\mathcal{U})$ . The definable family  $\{(a, b)_{\preccurlyeq} : a \in P, b \in Q\}$  is downward directed and so, by Lemma 5.2.6, extends to a definable type. This type must include  $\{\varphi(u, M^k) : u \in \Sigma\}$ . But then by construction  $c \notin \Sigma(\mathcal{U})$ , contradiction. This proves Claim 5.3.4.

We will need the following fact.

**Fact 5.3.5.** *For every choice of  $x_i \in P_i, y_i \in Q_i, 0 \leq i \leq m$ , there exists  $S \in \mathcal{S}$  such that  $S \subseteq \bigcup_i (x_i, y_i)_{\preccurlyeq}$  and is disjoint from every cut  $(P_j, Q_j)$ , for  $n < j \leq m$ .*

By Claim 5.3.4 this is witnessed by  $\mathbb{S}$  in  $\mathcal{U}$ , so it also holds in  $\mathcal{M}$ .

Finally we require a claim regarding non-definable cuts.

**Claim 5.3.6.** *Let  $(P, Q)$  be a non-definable cut. If  $F \subseteq X$  is a definable set such that, for every  $x \in P$ , there is  $y \in F$  with  $x \preccurlyeq y$ , then  $F \cap Q \neq \emptyset$ . Similarly if, for every  $x \in Q$ , there is  $y \in F$  with  $y \preccurlyeq x$ , then  $P \cap F \neq \emptyset$ .*

To prove Claim 5.3.6 note that, since  $F \subseteq X$  is definable, then its downward closure with respect to  $\preccurlyeq$ , namely  $F' = \{x \in X : x \preccurlyeq y \text{ for some } y \in F\}$ , is definable too. If  $F \cap P$  is cofinal in  $P$  then  $F' \supseteq P$ , so if  $P$  is not definable then it must be that  $F' \cap Q \neq \emptyset$ , hence  $F \cap Q \neq \emptyset$ . The case where  $P \cap Q$  is unbounded from below in  $Q$  is analogous. This proves Claim 5.3.6.

For any  $x_0, y_0, \dots, x_n, y_n \in X$  let  $rel(x_0, y_0, \dots, x_n, y_n)$  denote the formula, including parameters  $a_i, b_i$  for  $i > n$ , asserting that there exists  $S \in \mathcal{S}$  with

$$S \subseteq \bigcup_{0 \leq i \leq n} (x_i, y_i)_{\preccurlyeq} \cup \bigcup_{j > n} (a_j, b_j)_{\preccurlyeq}$$

such that  $S$  is disjoint from the cuts  $(P_j, Q_j)$  for  $n < j$ . Note that Fact 5.3.5 implies that  $rel(x_0, y_0, \dots, x_n, y_n)$  holds for every choice of  $x_i \in P_i$  and  $y_i \in Q_i$ ,  $0 < i \leq n$ . We define by reverse recursion a family of definable sets  $F_i, G_i \subseteq X$ , for  $0 < i \leq n$ , as follows. Let

$$F_n(x_0, y_0, \dots, x_{n-1}, y_{n-1}) = \{y \succcurlyeq a_n : rel(x_0, y_0, \dots, x_{n-1}, y_{n-1}, a_n, y)\} \text{ and}$$

$$G_n(x_0, y_0, \dots, x_{n-1}, y_{n-1}) = \{x \preccurlyeq b_n : rel(x_0, y_0, \dots, x_{n-1}, y_{n-1}, x, b_n)\}.$$

For  $0 < i < n$  let

$$F_i(x_0, y_0, \dots, x_{i-1}, y_{i-1}) = \{y \succcurlyeq a_i : \exists x' \in F_{i+1}(x_0, y_0, \dots, x_{i-1}, y_{i-1}, a_i, y)$$

$$\exists y' \in G_{i+1}(x_0, y_0, \dots, x_{i-1}, y_{i-1}, a_i, y), x' \preccurlyeq y'\}$$

and

$$G_i(x_0, y_0, \dots, x_{i-1}, y_{i-1}) = \{x \preccurlyeq b_i : \exists x' \in F_{i+1}(x_0, y_0, \dots, x_{i-1}, y_{i-1}, x, b_i)$$

$$\exists y' \in G_{i+1}(x_0, y_0, \dots, x_{i-1}, y_{i-1}, x, b_i), x' \preccurlyeq y'\}.$$

Clearly these sets are definable. For every  $0 < i \leq n$  we prove the following.

- (I<sub>i</sub>) For every choice of  $x_j \in P_j$ ,  $y_j \in Q_j$ ,  $0 \leq j < i$ , it holds that there exists  $x \in F_i(x_0, y_0, \dots, x_{i-1}, y_{i-1})$  and  $y \in G_i(x_0, y_0, \dots, x_{i-1}, y_{i-1})$  with  $x \preccurlyeq y$ .
- (II<sub>i</sub>) For every choice of  $a_j \preccurlyeq x_j \prec y_j \preccurlyeq b_j$ ,  $0 \leq j < i$ , if there exists  $x \in F_i(x_0, y_0, \dots, x_{i-1}, y_{i-1})$  and  $y \in G_i(x_0, y_0, \dots, x_{i-1}, y_{i-1})$  with  $x \preccurlyeq y$ , then there exists  $S \in \mathcal{S}$  with  $S \subseteq \bigcup_{j < i} (x_j, y_j)_{\preccurlyeq} \cup \bigcup_{j \geq i} (a_j, b_j)_{\preccurlyeq}$  that is disjoint from any cut  $(P_j, Q_j)$  for  $j \geq i$ .

We proceed by reverse induction on  $i$ , where in the inductive step (I<sub>i</sub>) follows from (I<sub>i+1</sub>) and (II<sub>i</sub>) from (II<sub>i+1</sub>). We will then derive the lemma from (I<sub>1</sub>) and (II<sub>1</sub>). Let  $i = n$ . Fix any  $x_j \in P_j$  and  $y_j \in Q_j$  for every  $j < n$ . By Fact 5.3.5 and Claim 5.3.6  $F_n(x_0, y_0, \dots, x_{n-1}, y_{n-1}) \cap P_n \neq \emptyset$  and  $G_n(x_0, y_0, \dots, x_{n-1}, y_{n-1}) \cap Q_n \neq \emptyset$ , so in particular there exists  $x \in F_n(x_0, y_0, \dots, x_{n-1}, y_{n-1})$  and  $y \in G_n(x_0, y_0, \dots, x_{n-1}, y_{n-1})$  with  $x \preccurlyeq y$ . This proves (I<sub>n</sub>). To prove (II<sub>n</sub>) let  $a_j \preccurlyeq x_j \prec y_j \preccurlyeq b_j$  for  $j < n$ . If there exists  $x \in F_n(x_0, y_0, \dots, x_{n-1}, y_{n-1})$  and  $y \in G_n(x_0, y_0, \dots, x_{n-1}, y_{n-1})$  with  $x \preccurlyeq y$  then it cannot

be that  $F_n(x_0, y_0, \dots, x_{n-1}, y_{n-1}) \subseteq Q_n$  and  $G_n(x_0, y_0, \dots, x_{n-1}, y_{n-1}) \subseteq P_n$ . Suppose that  $F_n(x_0, y_0, \dots, x_{n-1}, y_{n-1}) \cap P_n \neq \emptyset$ , the other case being similar. Then by definition of  $F_n$  there exists  $y \in P_n$  and some  $S \in \mathcal{S}$  with  $S \subseteq \cup_{0 \leq j < n} (s_j, t_j)_{\preceq} \cup (a_n, y)_{\preceq} \cup \cup_{j > n} (a_j, b_j)_{\preceq}$  such that  $F$  is disjoint from every cut  $(P_j, Q_j)$ ,  $j > n$ . By construction  $S$  is disjoint from  $(P_n, Q_n)$  too, proving (II<sub>n</sub>).

Suppose now that  $i < n$ . Fix  $x_j \in P_j$  and  $y_j \in Q_j$  for  $j < i$ . By (I<sub>i+1</sub>) and Claim 5.3.6 it holds that there exists  $x \in F_i(x_0, y_0, \dots, x_{i-1}, y_{i-1}) \cap P_i$  and  $y \in G_i(x_0, y_0, \dots, x_{i-1}, y_{i-1}) \cap Q_i$ , satisfying in particular  $x \preceq y$ . This proves (I<sub>i</sub>).

Now fix  $a_j \preceq x_j \prec y_j \preceq b_j$  for  $j < i$ . Once again if there are  $x \in F_i(x_0, y_0, \dots, x_{i-1}, y_{i-1})$  and  $y \in G_i(x_0, y_0, \dots, x_{i-1}, y_{i-1})$  with  $x \preceq y$  then it must be that either  $F_i(x_0, y_0, \dots, x_{i-1}, y_{i-1}) \cap P_i \neq \emptyset$  or  $G_i(x_0, y_0, \dots, x_{i-1}, y_{i-1}) \cap Q_i \neq \emptyset$ . We assume the former, being the proof given the latter analogous. Hence by definition of  $F_i$  there exists  $y \in P_i$  such that  $x' \preceq y'$  for some  $x' \in F_{i-1}(x_0, y_0, \dots, x_{i-1}, y_{i-1}, a_i, y)$ ,  $y \in G_{i-1}(x_0, y_0, \dots, x_{i-1}, y_{i-1}, a_i, y)$ . But then by (II<sub>i+1</sub>) there exists  $S \in \mathcal{S}$  such that  $S \subseteq \cup_{0 \leq j < i} (x_j, y_j)_{\preceq} \cup (a_i, y)_{\preceq} \cup \cup_{j > i} (a_j, b_j)_{\preceq}$ , and  $S$  is disjoint from any cut  $(P_j, Q_j)$  for  $j > i$ . However note that, by construction,  $S$  is also disjoint from the cut  $(P_i, Q_i)$ , which proves (II<sub>i</sub>).

Finally we derive the lemma from (I<sub>1</sub>), (II<sub>1</sub>) and Claim 5.3.6 by repeating the arguments in the inductive step. That is, by (I<sub>1</sub>) and Claim 5.3.6 there exists some  $y \in P_0$  such that  $x' \preceq y'$  for some  $x' \in F_1(a_0, y)$ ,  $y' \in G_1(a_0, y)$ . Then by (II<sub>1</sub>) there is some  $S \in \mathcal{S}$  with  $S \subseteq (a_0, y)_{\preceq} \cup \cup_{j > 0} (a_j, b_j)_{\preceq}$  that is disjoint from any cut  $(P_j, Q_j)$  for  $j > 0$ . By construction  $S$  is also disjoint from  $(P_0, Q_0)$ .  $\square$

By letting  $(X, \preceq) = (M, \leq)$ , Lemma 5.3.3 proves the case  $\mathcal{S} \subseteq \mathcal{P}(M)$  of Theorem 5.3.9, i.e. it shows that, if  $\mathcal{S}$  is a definable family of nonempty sets with the  $(m, 2)$ -property for any  $m \geq 2$ , then it has a finite tame transversal. This result can be generalized to higher dimensions as follows.

**Proposition 5.3.7.** *Let  $\mathcal{S} \subseteq \mathcal{P}(M^n)$  be a definable family of nonempty sets. At least one of the following holds.*

- (1) *There exists an infinite subfamily of  $\mathcal{S}$  of pairwise disjoint sets.*

(2) *There exists a tame transversal  $T$  for  $\mathcal{S}$  with  $\dim T < n$ .*

*Proof.* We assume the negation of (1) and prove (2). We work by induction on  $n$ . The case  $n = 1$  is given by Lemma 5.3.3. Suppose that  $n > 1$ . Let  $\pi_1$  denote the projection onto the first coordinate and consider the definable family  $\pi_1(\mathcal{S}) = \{\pi_1(S) : S \in \mathcal{S}\}$ . If  $\mathcal{S}$  does not contain an infinite subfamily of pairwise disjoint sets then the same holds for  $\pi_1(\mathcal{S})$ . By the case  $n = 1$  there exists a finite set  $\{\xi_0, \dots, \xi_m\}$  that is a tame transversal for  $\pi_1(\mathcal{S})$ . Let  $\mathcal{N} = (N, <, \dots)$  be a tame extension of  $\mathcal{M}$  that contains  $\{\xi_0, \dots, \xi_m\}$ . Then  $\cup_{0 \leq i \leq m} \{\xi_i\} \times N^{n-1}$  is a tame transversal for  $\mathcal{S}$  of dimension  $n - 1$ .  $\square$

In order to prove Theorem 5.3.9 we first show that any definable family with the finite intersection property has a finite tame transversal. We in fact prove the following more precise statement. Recall that a family of sets  $\mathcal{S}$  is  $n$ -consistent if any subfamily of at most  $n$  sets has nonempty intersection.

**Lemma 5.3.8.** *Let  $\mathcal{S} \subseteq \mathcal{P}(M^n)$  be a definable family of sets. If  $\mathcal{S}$  is  $2^n$ -consistent then it admits a finite tame transversal.*

*Proof.* We proceed by induction on  $n$ . The case  $n = 1$  is given by Lemma 5.3.3. Suppose  $n > 1$ . Let  $\mathcal{S} = \{\varphi(u, M^n) : u \in \Omega\}$ .

Consider the family  $\{\pi(S \cap S') : S, S' \in \mathcal{S}\}$ . This family is definable and  $2^{n-1}$ -consistent. Hence by induction hypothesis there exists a finite collection of definable  $(n - 1)$ -types  $p_0, \dots, p_m$  such that, for every  $S, S' \in \mathcal{S}$ ,  $\pi(S \cap S')$  belongs in one of them. We construct a finite tame transversal for  $\mathcal{S}$  given by types whose projection is one of  $p_0, \dots, p_m$ . The approach is to use cell decomposition to witness the “fiber over the  $p_i$ ” of each set in  $\mathcal{S}$  as a finite union of points and intervals on some definable preordered set, and then apply Lemma 5.3.3.

Let  $\mathcal{D}$  be a uniform cell decomposition of  $\mathcal{S}$ , i.e. a cell decomposition of  $\varphi(\Omega, M^n)$ . For any  $D \in \mathcal{D}$  and  $u \in \Omega$  with  $D_u \neq \emptyset$  let  $f_{D_u}$  and  $g_{D_u}$  denote the functions such that  $D_u = (f_{D_u}, g_{D_u})$  or  $D_u = \text{graph}(f_{D_u}) = \text{graph}(g_{D_u})$ . Let  $\mathcal{H}_i = \{f_{D_u}, g_{D_u} : D \in \mathcal{D}, u \in \Omega, \pi(D_u) \in p_i\}$ . We briefly observe that this family is definable.

For every  $i$  and  $D \in \mathcal{D}$  let  $\Omega(i, D) = \{u \in \Omega : \pi(D_u) \in p_i\}$ . By definability of  $p_i$  these sets are definable. By re-indexing  $\mathcal{H}_i$  in terms of a finite disjoint union  $X_i$  of sets  $\Omega(i, D)$  for  $D \in \mathcal{D}$ , we may take  $\mathcal{H}_i$  to be a definable family  $\{h_x : x \in X_i\}$ .

Let  $\preceq_i$  be the total preorder induced by  $p_i$ , in particular on  $\mathcal{H}_i$ , described in Section 5.1.1. We abuse notation and also let  $\preceq_i$  denote the induced definable preorder on the index set  $X_i$ . Let  $(X, \preceq)$  be the definable totally preordered set that extends the disjoint union of  $(X_i, \preceq_i)$ ,  $0 \leq i \leq m$ , by letting  $x \prec y$  for any  $x \in X_i, y \in X_j$  with  $i < j$ .

For every  $u \in \Omega$ , let  $B_u$  denote the union in  $(X, \preceq)$  of intervals  $[y, z]_{\preceq}$  where  $y, z \in X_i$  for some  $0 \leq i \leq m$  and satisfy that there exists  $D \in \mathcal{D}$  with  $D_u = (h_y, h_z)$  or  $D_u = \text{graph}(h_y) = \text{graph}(h_z)$ . So every set  $B_u$  is the union of at most  $(m+1)|\mathcal{D}|$  closed intervals. By definability of  $\mathcal{S}, \preceq$  and each  $X_i$  the family  $\mathcal{B}_{\mathcal{S}} = \{B_u : u \in \Omega\}$  is definable. Onwards for clarity we write  $B_S$  in place of  $B_u$  where  $S = \varphi(u, M^n)$ . This is valid because  $B_u = B_v$  whenever  $S = \varphi(u, M^n) = \varphi(v, M^n)$ , although this observation is unnecessary since we could complete the proof in terms of the subscript  $u$ .

Since every two sets  $S, S' \in \mathcal{S}$  satisfy that  $\pi(S \cap S') \in p_i$  for some  $0 \leq i \leq m$ , the family  $\mathcal{B}_{\mathcal{S}}$  is 2-consistent. Consequently, by Lemma 5.3.3, there exist finitely many definable types  $q_0, \dots, q_l$  such that, for every  $S \in \mathcal{S}$ , the set  $B_S$  belongs in one of them. We complete the proof by fixing an arbitrary  $0 \leq j \leq l$  and showing that the subfamily  $\mathcal{S}_j = \{S \in \mathcal{S} : B_S \in q_j\}$  admits a finite tame transversal. We will use the construction of types appearing in Definition 5.2.13.

Let  $i$  be such that  $X_i \in q_j$ . Note that, for every  $S \in \mathcal{S}$ , if  $B_S \in q_j$  then there exists  $y, z \in X_i$  with  $[y, z]_{\preceq} \in q_i$ , such that  $h_y$  and  $h_z$  are involved in the definition of a cell inside  $S$ , i.e.  $\text{dom}(h_y) = \text{dom}(h_z)$  and  $(h_y, h_z) \subseteq S$  or  $\text{graph}(h_y) = \text{graph}(h_z) \subseteq S$ . Hence, by definition of  $\preceq$ , for every  $x \in X_i$ , if  $y \preceq x \preceq z$  then it must hold that either  $S \in h_x|_{p_i}$ ,  $S \in h_x^+|_{p_i}$  or  $S \in h_x^-|_{p_i}$ .

We define  $Q = \{x \in X_i : (-\infty, x]_{\preceq} \in p_i\}$ . Note that, by definability of  $p_i$ , this set is definable. Let  $S \in \mathcal{S}_j$  and  $y, z \in X_i$  with  $[y, z]_{\preceq} \in q_i$  be such that  $h_y$  and  $h_z$  are involved in the definition of a cell inside  $S$ . Then clearly  $z \in Q$ , and moreover  $[y, z]_{\preceq} \cap (-\infty, x]_{\preceq} \subseteq [y, x]_{\preceq} \in q_i$  for every  $x \in Q$ , so  $y \preceq Q$ . If  $Q$  has a minimum  $\hat{x}$  it follows that  $y \preceq \hat{x} \preceq z$ , and so by the above paragraph the family  $\mathcal{S}_j$  has a tame transversal of size at most three given by the



types  $h_{\hat{x}}|_{p_i}$ ,  $h_{\hat{x}}^+|_{p_i}$  and  $h_{\hat{x}}^-|_{p_i}$ . Finally, we assume that  $Q$  does not have a minimum. In this case it holds that  $y \prec Q$  and  $x \prec z$  for some  $x \in Q$ . Note that, by definition of  $\prec$ , the inequality  $y \prec x \prec z$  implies  $S \in h_x|_{p_i}$ . We show that in this case  $\mathcal{S}_j$  extends to a definable type.

Let  $q$  be the type of all sets  $S \in Def(M^n)$  such that  $S \in h_x|_{p_i}$  for all  $x$  in some subset of  $Q$  that is unbounded from below. We show consistency of this type. By cell decomposition it suffices to do it for cells. Let  $C_1, C_2$  be two cells in  $q$ . For  $k \in \{1, 2\}$  it must be that  $C_k = (f_k, g_k)$  where  $f_k \prec_i \{h_x : x \in Q\}$  and  $h_{x_k} \preceq_i f_k$  for some  $x_k \in Q$ . If  $x = \min_{\prec} \{x_1, x_2\}$  then we have that  $C_1 \cap C_2 \in h_{x'}|_{p_i}$  for every  $x' \in Q$  with  $x' \prec x$ . So  $C_1 \cap C_2 \in q$ . Note that, by definability of  $p_i$ ,  $\mathcal{H}_i$  and  $Q$ , the type  $q$  is definable. In the previous paragraph we have observed that  $\mathcal{S}_j \subseteq q$ . This completes the proof of the lemma.  $\square$

We may now prove Theorem 5.3.9. Recall that a family of sets is  $n$ -inconsistent if every subfamily of size  $n$  has empty intersection. Recall also that, for a definable set  $X$ , we denote its boundary (in the euclidean topology) by  $bd(X)$ . If  $X$  is contained in a set  $Y$  then let  $bd_Y(X) = bd(X) \cap Y$ .

**Theorem 5.3.9.** *Let  $\mathcal{S} \subseteq \mathcal{P}(M^k)$  be a definable family of nonempty sets. Let  $n = \max\{1, \dim \cup \mathcal{S}\}$ . The following are equivalent.*

- (1) *There exists some  $m \geq n + 1$  such that  $\mathcal{S}$  has the  $(m, n + 1)$ -property.*
- (2)  *$\mathcal{S}$  has a finite tame transversal.*

*Proof.* To prove that (2) implies (1) note that, if  $\mathcal{S}$  admits a tame transversal of cardinality  $l$ , then it has a covering of  $l$  subfamilies with the FIP (those given by the sets containing a given element in the transversal). It follows that  $\mathcal{S}$  has the  $(nl + 1, n + 1)$ -property. We assume the negation of (2) and derive the negation of (1). We do so by proving the following apparently stronger statement.

Suppose that  $\mathcal{S} \subseteq \mathcal{P}(M^k)$  does not admit a finite tame transversal. Let  $m$  and  $n$  be such that  $m > n \geq \max\{1, \dim \cup \mathcal{S}\}$ . Let  $X_1 \subseteq \dots \subseteq X_n \subseteq X_{n+1}$  be a collection of nested definable sets satisfying that  $\dim X_i < i$  for every  $1 \leq i \leq n$  and  $\cup \mathcal{S} \subseteq X_{n+1}$ . Then there exists  $\mathcal{F} \subseteq \mathcal{S}$  with  $|\mathcal{F}| = m$  such that, for every  $1 \leq i \leq n + 1$  and subfamily  $\mathcal{F}' \subseteq \mathcal{F}$  with

$|\mathcal{F}'| = i$ , it holds that  $\cap \mathcal{F}' \cap X_i = \emptyset$ . Namely, for every  $1 \leq i \leq n+1$ , the family  $\mathcal{F} \cap X_i$  is  $i$ -inconsistent. In particular, since  $\cup \mathcal{S} \subseteq X_{n+1}$ , the family  $\mathcal{F}$  is  $(n+1)$ -inconsistent, and so  $\mathcal{S}$  does not have the  $(m, n+1)$ -property.

Let  $Prop(m, n)$  denote the statement that the above holds for some fixed  $m > n \geq 1$  and any  $\mathcal{S}$  that does not have a finite tame transversal and moreover satisfies that  $n \geq \dim \cup \mathcal{S}$ . We prove  $Prop(m, n)$  by induction. We let  $Prop(2, 1)$  be the base case. Suppose that  $m > 2$ . If  $m > n+1$  then we use  $Prop(m-1, n)$  to derive  $Prop(m, n)$ . Otherwise we use  $Prop(m-1, n-1)$  to derive  $Prop(m, n)$ . We prove all cases simultaneously. We divide the argument into two parts.

Let us fix  $m > n \geq 1$  and  $\mathcal{S} \subseteq \mathcal{P}(M^k)$  a definable family of nonempty sets with  $n \geq \dim \cup \mathcal{S}$  that does not admit a finite tame transversal, and  $X_1 \subseteq \dots \subseteq X_{n+1}$  a family of definable nested sets with  $\dim X_i < i$  for  $i \leq n$  and  $\cup \mathcal{S} \subseteq X_{n+1}$ . The first part of the argument consists of the following claim.

**Claim 5.3.10.** *Let  $\mathcal{S}' \subseteq \mathcal{S}$  be a finite subfamily and, for every  $1 \leq i \leq n$ , let  $Y_i = \cup_{S \in \mathcal{S}'} bd_{X_{i+1}}(S \cap X_{i+1})$ . There exists a subfamily  $\mathcal{F} \subseteq \mathcal{S}$  of cardinality  $m-1$  that is  $(n+1)$ -inconsistent such that  $\mathcal{F} \cap (X_i \cup Y_i)$  is  $i$ -inconsistent for  $1 \leq i \leq n$ .*

Note that, by o-minimality,  $\dim Y_i < i$  for every  $i$ . So  $\dim(X_i \cup Y_i) < i$  for every  $1 \leq i \leq n$ . Note that, if  $m = n+1$ , then any subfamily  $\mathcal{F} \subseteq \mathcal{S}$  of cardinality  $m-1 = n$  is  $(n+1)$ -inconsistent vacuously. It follows that, if  $m = 2$  and  $n = 1$ , the claim follows easily from the fact that  $\mathcal{S}$  does not admit a finite transversal in  $M^k$ .

Suppose that  $m > 2$ . If  $m > n+1$  then the claim is given by  $Prop(m-1, n)$  applied to  $\mathcal{S}$  and the collection of nested sets  $\{X_1 \cup Y_1, \dots, X_n \cup Y_n, X_{n+1}\}$ . Suppose  $m = n+1 > 2$ . We may apply  $Prop(m-1, n-1) = Prop(n, n-1)$  to the definable family  $\mathcal{S} \cap (X_n \cup Y_n)$  and collection of nested sets  $\{X_1 \cup Y_1, \dots, X_n \cup Y_n\}$ . It follows that there exists  $\mathcal{F} \subseteq \mathcal{S}$  with cardinality  $m-1 = n$  such that  $\mathcal{F} \cap (X_n \cup Y_n) \cap (X_i \cup Y_i) = \mathcal{F} \cap (X_i \cup Y_i)$  is  $i$ -inconsistent for  $1 \leq i \leq n$ . Since  $|\mathcal{F}| = n$ , the family  $\mathcal{F}$  is  $(n+1)$ -inconsistent vacuously. This completes the proof of the claim.

We continue with the second part of the proof.

For any  $S \in \mathcal{S}$  consider the family of all ordered tuples  $\langle S_1, \dots, S_{m-1} \rangle \in \prod_{1 \leq i < m} \mathcal{S}$  such that  $\{S_1, \dots, S_{m-1}\}$  is  $(n+1)$ -inconsistent and  $\{S_1, \dots, S_{m-1}\} \cap (X_i \cup \text{bd}_{X_{i+1}}(S \cap X_{i+1}))$  is  $i$ -inconsistent for every  $1 \leq i \leq n$ . We denote this family  $\mathcal{S}^m(S)$ . Note that, by Claim 5.3.10, the family  $\{\mathcal{S}^m(S) : S \in \mathcal{S}\}$  has the finite intersection property.

We now explicitly identify the family  $\{\mathcal{S}^m(S) : S \in \mathcal{S}\}$  with a family of definable sets as follows. Let  $\varphi(u, v)$  be such that  $\mathcal{S} = \{\varphi(u, M^k) : u \in \Omega\}$ . For any  $u \in \Omega$ , with  $S = \varphi(u, M^k)$ , let  $\Omega^m(u)$  denote the set of all index tuples  $\langle u_1, \dots, u_{m-1} \rangle$  such that  $\langle \varphi(u_1, M^k), \dots, \varphi(u_{m-1}, M^k) \rangle \in \mathcal{S}^m(S)$ . Clearly the family  $\{\Omega^m(u) : u \in \Omega\}$  is definable. Like  $\{\mathcal{S}^m(S) : S \in \mathcal{S}\}$ , it has the finite intersection property.

By Lemma 5.3.8 it follows that  $\{\Omega^m(u) : u \in \Omega\}$  admits a finite tame transversal. That is, we may partition  $\Omega$  into finitely many definable subfamilies  $\Omega_1, \dots, \Omega_s$  satisfying that, for each  $1 \leq i \leq s$ , the family  $\{\Omega^m(u) : u \in \Omega_i\}$  extends to a definable type. Note that, since by assumption  $\mathcal{S}$  does not have a finite tame transversal, there must be some  $1 \leq i \leq s$  such that  $\mathcal{S}_i = \{\varphi(u, M^k) : u \in \Omega_i\}$  does not have a finite tame transversal either. Hence, by passing if necessary to a subfamily of  $\mathcal{S}$ , we may assume that  $\{\Omega^m(u) : u \in \Omega\}$  extends to a definable type  $p$ .

Let  $\mathcal{N} = (N, \dots)$  be a tame extension of  $\mathcal{M}$  that realizes  $p$ . Such an extension exists by the Marker-Steinhorn Theorem (Theorem 2.1.8). Onwards let  $S^* = S(\mathcal{N})$  and  $X_i^* = X_i(\mathcal{N})$  for every  $S \in \mathcal{S}$  and  $1 \leq i \leq n+1$ . That is, we use an asterisk to denote the interpretation in  $\mathcal{N}$  of a definable set in  $\mathcal{M}$ . Let  $c = \langle c_1, \dots, c_{m-1} \rangle$  be a realization in  $\mathcal{N}$  of  $p$  and, for  $1 \leq i < m$ , let  $S_i = \varphi(c_i, N^k)$ . Then we have that  $\{S_1, \dots, S_{m-1}\}$  is  $(n+1)$ -inconsistent and  $\{S_1, \dots, S_{m-1}\} \cap (X_i^* \cup \text{bd}_{X_{i+1}^*}(S^* \cap X_{i+1}^*))$  is  $i$ -inconsistent for every  $1 \leq i \leq n$ . We prove that there exists  $S \in \mathcal{S}$  such that the family  $\{S_1, \dots, S_{m-1}, S^*\}$  witnesses  $\text{Prop}(m, n)$  in  $\mathcal{N}$ , by satisfying that  $\{S_1, \dots, S_{m-1}, S^*\} \cap X_i^*$  is  $i$ -inconsistent for every  $1 \leq i \leq n+1$ . Since  $\mathcal{N}$  is an elementary extension then an analogous family witnessing  $\text{Prop}(m, n)$  in  $\mathcal{M}$  must exist too.

Let  $\mathcal{C}$  be a cell decomposition of  $N^k$  compatible with each intersection of sets in  $\{S_1, \dots, S_{m-1}, X_1^*, \dots, X_n^*, X_{n+1}^*\}$ . For each cell  $C \in \mathcal{C}$  we choose a point  $\xi_C \in C$ . Since  $\mathcal{S}$  does not have a finite tame transversal there exists  $S \in \mathcal{S}$  such that  $S^* \cap \{\xi_C : C \in \mathcal{C}\} = \emptyset$ .

Since  $X_1$  is finite we also choose  $S$  so that  $S \cap X_1 = \emptyset$ . Let  $\mathcal{F} = \{S_1, \dots, S_{m-1}, S^*\}$ . We show that  $\mathcal{F} \cap X_i^*$  is  $i$ -inconsistent for every  $1 \leq i \leq n+1$ .

Towards a contradiction suppose that there exists some  $1 \leq i \leq n+1$  and a subset  $\mathcal{F}' \subseteq \mathcal{F}$  of cardinality  $i$  such that  $\cap \mathcal{F}' \cap X_i^* \neq \emptyset$ . Note that, by construction of  $\{S_1, \dots, S_{m-1}\}$ , it must be that  $S^* \in \mathcal{F}'$ . If  $i = 1$  then  $\mathcal{F}' = \{S^*\}$  and  $S^* \cap X_1^* \neq \emptyset$  contradicts that  $S \cap X_1 = \emptyset$ . Suppose that  $i > 1$ . Let  $\mathcal{F}'' = \mathcal{F}' \setminus \{S^*\}$  and let us fix a cell  $C \in \mathcal{C}$  such that  $C \cap (\cap \mathcal{F}') \cap X_i^* \neq \emptyset$ . Since  $\mathcal{C}$  is compatible with  $\cap \mathcal{F}'' \cap X_i^*$  it must be that  $C \subseteq \cap \mathcal{F}'' \cap X_i^*$ . Moreover recall that  $\cap \mathcal{F}''$  (and in particular  $C$ ) is disjoint from  $bd_{X_i^*}(S^* \cap X_i^*)$ . By definable connectedness from  $C \cap S^* \neq \emptyset$  it follows that  $C \subseteq S^*$ . This however contradicts that  $\xi_C \notin S^*$ . This completes the proof of the theorem.  $\square$

Note that, if we add to the statement of Theorem 5.3.9 the condition that all the sets in  $\mathcal{S}$  are finite, then the transversal can always be assumed to be in  $M$ .

With the use of the Alon-Kleitman-Matoušek  $(p, q)$ -theorem (Theorem 5.1.5), Lemma 5.3.8 can be used to prove a different version of Theorem 5.3.9, where the  $n+1$  in the statement is substituted by any interger greater than the VC-codensity of  $\mathcal{S}$ . We highlight this in the following corollary.

**Corollary 5.3.11.** *Let  $\mathcal{S}$  be a definable family of sets with the  $(m, n)$ -property, where  $m \geq n$  and  $n$  is greater than the VC-codensity of  $\mathcal{S}$ . Then  $\mathcal{S}$  admits a finite tame transversal.*

*Proof.* For simplicity we assume that  $\mathcal{S} \subseteq \mathcal{P}(M)$ , the proof otherwise being analogous. Applying Theorem 5.1.5 there exists a natural number  $k > 0$  such that every finite  $\mathcal{S}' \subseteq \mathcal{S}$  has a transversal of cardinality at most  $k$ .

Let  $\mathcal{F} = \{F(S) \subseteq M^k : S \in \mathcal{S}\}$  be the definable family of sets given by  $F(S) = (S \times M^{k-1}) \cup (M \times S \times M^{k-2}) \cup \dots \cup (M^{k-1} \times S)$ . Note that  $\mathcal{F}$  has the FIP. To see this it suffices to note that, if  $\mathcal{S}'$  is a finite subfamily of  $\mathcal{S}$ , and  $\{x_1, \dots, x_k\} \subseteq M^k$  is a transversal for  $\mathcal{S}'$ , then  $\langle x_1, \dots, x_k \rangle \in M^k$  is in every  $F(S)$  for  $S \in \mathcal{S}'$ .

So, applying Lemma 5.3.8,  $\mathcal{F}$  has a finite tame transversal  $G \subseteq U^k$ . Let  $H = \cup_{1 \leq i \leq k} \pi_i(G)$ , where  $\pi_i$  denotes the projection to the  $i$ -th coordinate. We claim that  $H$  is a tame transversal for  $\mathcal{S}$ . This follows from the observation that, for every  $S \in \mathcal{S}$  and  $\langle x_1, \dots, x_k \rangle \in F(S)$ , there is some  $i$  such that  $x_i \in S$ .  $\square$

Recall that Theorem 5.1.6 states that any uniform family  $\mathcal{S} \subseteq \text{Def}(M^n)$  has VC-codensity at most  $n$ . One may use this to derive Theorem 5.3.9 from Corollary 5.3.11.

The following example shows that the bounds in Theorem 5.3.9 and Corollary 5.3.11 cannot be improved.

**Example 5.3.12.** For any  $u, v \in M$  consider the “cross”  $S(u, v) = (\{u\} \times M) \cup (M \times \{v\})$ . The family of crosses  $\mathcal{S} = \{S(u, v) : u, v \in M\}$  is a definable family of subsets of  $M^2$  with VC-codensity 2. To prove the latter we leave it to the reader to check that  $\binom{n}{2} + n \leq \pi_{\mathcal{S}}^*(n) \leq (n+1)^2$ . Moreover  $\mathcal{S}$  is 2-consistent, since  $\{\langle u_1, v_2 \rangle, \langle u_2, v_1 \rangle\} \subseteq S(u_1, v_1) \cap S(u_2, v_2)$  for every  $u_1, v_1, u_2, v_2 \in M$ . We observe that  $\mathcal{S}$  does not have a finite transversal in any elementary extension.

Let  $\mathcal{M} \preceq \mathcal{N} = (N, \dots)$  and  $X \subseteq N^2$  be a finite set. Let  $X' \subseteq M$  be the set of coordinates of points in  $X$ . Pick any  $u, v \notin X' \cap M$ . Then clearly the interpretation of  $S(u, v)$  in  $\mathcal{N}$  is disjoint from  $X$ .

Note that Lemma 5.3.3 shows that a definable family of subsets of  $M$  has a finite tame transversal if and only if it has the  $(\omega, 2)$ -property. We ask whether Theorem 5.3.9 can be improved to generalize this to higher dimensions.

**Question 5.3.13.** *Let  $\mathcal{S} \subseteq (M^n)$  be a definable family of nonempty sets and let  $n = \max\{1, \max \cup \mathcal{S}\}$ . If  $\mathcal{S}$  has the  $(\omega, n+1)$  property then does it have a finite tame transversal?*

Note that the answer to the above question is positive when  $\mathcal{M}$  is  $\omega_1$ -saturated.

We now present a proposition that follows from the ideas in the proof of Lemma 5.3.3.

**Proposition 5.3.14.** *Let  $\mathcal{S} \subseteq \mathcal{P}(M)$  and  $k$  be such that every  $S \in \mathcal{S}$  is union of at most  $k$  open or closed intervals and points. Suppose that  $\mathcal{S}$  is  $(k+1)$ -consistent. Then at least one of the following holds:*

- (i)  $\mathcal{S}$  has a finite transversal in  $M$ ,
- (ii)  $\mathcal{S}$  extends to a definable type.

*Proof.* Suppose that  $\mathcal{S}$  as in the proposition does not have a finite transversal in  $M$  and also does not extend to a definable type. We prove that  $\mathcal{S}$  is not  $(k+1)$ -consistent. Since the

construction and arguments that follow are similar to those in the proof of Lemma 5.3.3 we are concise in the presentation.

If  $\mathcal{S}$  does not have a finite transversal in  $M$  then for every finite  $X \subseteq M$  there is  $S \in \mathcal{S}$  with  $S \cap X = \emptyset$ . Let  $\mathbb{S} \in \mathcal{S}(\mathcal{U})$  be such that  $\mathbb{S} \cap M = \emptyset$ . Since  $\mathbb{S}$  is union of at most  $k$  points and intervals, there are at most  $k$  cuts  $(P_1, Q_1), \dots, (P_m, Q_m)$  in  $M$  such that every point in  $\mathbb{S}$  realizes one of these cuts. We show that for every  $1 \leq i \leq m$  there exists some  $S_i \in \mathcal{S}$  that is disjoint from  $(P_i, Q_i)$ . It follows that the family  $\mathbb{S}, S_1(\mathcal{U}), \dots, S_m(\mathcal{U})$ , which has size at most  $k + 1$ , has empty intersection. Hence  $\mathcal{S}$  is not  $(k + 1)$ -consistent.

Let us fix  $1 \leq i \leq m$ . If  $(P_i, Q_i)$  is definable then by assumption there exists some  $S_i \in \mathcal{S}$  that is disjoint from the cut. Suppose that  $(P_i, Q_i)$  is not definable. Let us fix  $a \in P_i$  and  $b \in Q_i$  with  $\{a, b\} \subset P_j$  or  $\{a, b\} \subset Q_j$  for every  $j \neq i$ . Note that, since  $(P_i, Q_i)$  is not definable, we have that  $(a, +\infty) \cap P_i \neq \emptyset$ . Consider the definable set  $F = \{t \in M : t > a \text{ and there exists } S \in \mathcal{S} \text{ with } (a, t) \cap S = \emptyset\}$ . Then  $\mathbb{S}$  witnesses the fact that  $(a, +\infty) \cap P_i \subseteq F$ . Since  $(P_i, Q_i)$  is not definable we have that  $F \cap Q_i \neq \emptyset$  (see Claim 5.3.6 in the proof of Lemma 5.3.3). By definition of  $F$  we conclude that there exists some  $S_i \in \mathcal{S}$  disjoint from  $(P_i, Q_i)$ .  $\square$

It is possible that the above proposition can be generalized to higher dimensions, albeit that author has not been able to prove a precise statement for said generalization.

We end with another example of a result on transversals from finite and compact combinatorics (Proposition 5.1.4) that adapts to definable families in o-minimal structures. Since the result is not central to this chapter, we will, like in the previous proposition, be concise in the proof.

**Proposition 5.3.15.** *Let  $\mathcal{S} \subseteq \mathcal{P}(M)$  be a definable family of intervals. Let  $k \geq 1$  be the maximum such that there exists  $k$  pairwise disjoint sets in  $\mathcal{S}$ . Then  $\mathcal{S}$  has a tame transversal of size  $k$ .*

*Proof.* We proceed by induction on  $k$ .

For the base case  $k = 1$  let  $H$  be the definable set of all left endpoints of intervals in  $\mathcal{S}$ . Let  $a = \sup H$ . Since  $k = 1$ , meaning that  $\mathcal{S}$  is 2-consistent, note that, by definition of  $H$ , every right endpoint of an interval in  $\mathcal{S}$  must be greater or equal to  $a$ . Now suppose that

$\mathcal{S}$  does not extend to the definable type with basis  $\{(t, a) : t < a\}$ . Then there must exist  $S \in \mathcal{S}$  with  $S \subseteq [a, +\infty)$ . Additionally suppose that  $\mathcal{S}$  does not extend to the definable type with basis  $\{(a, t) : a < t\}$ . Then there must exist  $S' \in \mathcal{S}$  with  $S' \subseteq (-\infty, a]$ . Finally, suppose that  $\mathcal{S}$  does also not extend to  $\text{tp}(a/M)$ , and let  $S'' \in \mathcal{S}$  be such that  $a \notin S''$ . If  $S'' \subseteq (-\infty, a)$  then  $S'' \cap S = \emptyset$  and if  $S'' \subseteq (a, +\infty)$  then  $S'' \cap S' = \emptyset$ , contradicting that  $k = 1$ .

Now let  $k > 1$ . Let  $\mathcal{S}_0 \subseteq \mathcal{S}$  be the subfamily of leftmost intervals in any pairwise disjoint subfamily of  $\mathcal{S}$  of size  $k$ . Note that, by induction hypothesis, the definable family  $\mathcal{S} \setminus \mathcal{S}_0$  has a tame transversal of size  $k - 1$ . We complete the proof by noting that  $\mathcal{S}_0$  is 2-consistent, and so, by the base case, extends to a definable type.

Suppose towards a contradiction that there are  $S, S' \in \mathcal{S}_0$  with  $S \cap S' = \emptyset$ . Without loss of generality let  $S < S'$ . Let  $\mathcal{F} \subseteq \mathcal{S}$  be a subfamily of size  $k$  of pairwise disjoint sets that includes  $S'$  as leftmost interval. Then the family  $\{S\} \cup \mathcal{F}$  is a family of size  $k + 1$  of pairwise disjoint sets. Contradiction.  $\square$

### 5.3.1 Forking, dividing and definable types

In this subsection we reformulate Theorem 5.3.9 as a statement about the relationship between forking and definable types known in o-minimal and some more general NIP theories. This is the subject of ongoing research among NIP theories [46]. Until the end of the subsection we drop the assumption that  $\mathcal{M}$  and  $\mathcal{U}$  are o-minimal.

Recall that formula  $\varphi(x, b)$  *k-divides over*  $A$  (a small set), for some  $k \geq 1$ , if there exists a sequence of elements  $(b_i)_{i < \omega}$  in  $U^{l(b)}$ , with  $\text{tp}(b_i/A) = \text{tp}(b/A)$ , such that every  $k$ -element subset of  $\{\varphi(x, b_i) : i < \omega\}$  is inconsistent. Equivalently, if the family of sets of the form  $\varphi(U^{l(x)}, b')$ , where  $\text{tp}(b'/A) = \text{tp}(b/A)$ , does not have the  $(\omega, k)$ -property. A formula  $\varphi(x, b)$  *divides* if it *k-divides* for some  $k$ . Conversely, a formula  $\varphi(x, b)$  does not divide over  $A$  if the family  $\{\varphi(U^{l(x)}, b') : \text{tp}(b'/A) = \text{tp}(b/A)\}$  has the  $(\omega, k)$ -property for every  $k$ . Hence, not dividing is an intersection property.

A formula *forks over*  $A$  if it implies a finite disjunction of formulas that divide each over  $A$ . In  $NTP_2$  theories (a class which includes NIP and simple theories) forking and dividing over a model are equivalent notions (see Theorem 1.1 in [8]).

The next equivalence was proved first for o-minimal expansions of fields<sup>2</sup> by Dolich [12] (where he considers forking over any set and not just models) and for unpackable VC-minimal theories (a class which includes o-minimal theories) by Cotter and Starchenko [9]. The following is the best generalization up to date, due to Simon and Starchenko [47].

**Theorem 5.3.16** ([47], Theorem 5). *Let  $T$  be a dp-minimal  $\mathcal{L}$ -theory with monster model  $\mathcal{U}$  satisfying that, for every set  $A \subseteq U$ , every unary set definable over  $A$  extends to a definable type in  $S_1(A)$ . Let  $M \models T$  and  $\varphi(x, d) \in \mathcal{L}(U)$ . The following are equivalent.*

- (i)  $\varphi(x, d)$  does not fork over  $M$ .
- (ii)  $\varphi(x, d)$  extends to an  $M$ -definable type.

For precise definitions of unpackable VC-minimal and dp-minimal theory see the corresponding references. By the equivalence between forking and dividing in  $NTP_2$  theories, (i) and (ii) above are also equivalent to  $\varphi(x, d)$  not dividing over  $M$ .

We wish to observe that, in the o-minimal setting, a version of Theorem 5.3.16, which considers any set  $A \subseteq M$  in place of  $M$  and a condition weaker than not dividing, follows from our Theorem 5.3.9. First however we must prove a parameter characterization of the property of having a finite tame transversal. This corresponds to Proposition 5.3.18. We first need a short lemma.

**Lemma 5.3.17.** *Let  $A \subseteq M$ . Let  $p \in S_n(U)$  be a type definable over a tuple  $b \in U^m$ . Suppose the  $\text{tp}(b/M)$  is  $A$ -definable. Then the restriction of  $p$  to  $M$  (the partial type of  $M$ -definable sets in  $p$ ) is  $A$ -definable.*

*Proof.* Let  $\varphi(x, y)$  be a partitioned formula with  $l(x) = n$  and  $\psi(y, b)$  be the definition of  $p|_\varphi$ . Since the type of  $b$  over  $M$  is  $A$ -definable the set  $\psi(M^n, b)$ , which corresponds to all  $y \in M^{l(y)}$  such that  $\varphi(x, y) \in p$ , is  $A$ -definable.  $\square$

<sup>2</sup>↑Dolich actually works with “nice” o-minimal theories, a class of structures which includes o-minimal expansions of fields.



**Proposition 5.3.18** (Parameter characterization of having FTT). *Let  $\mathcal{M}$  be o-minimal and  $A \subseteq M$ . Let  $\mathcal{S}$  be an  $A$ -definable family of sets with a finite tame transversal of size  $n$ . Then  $\mathcal{S}$  has a finite tame transversal over  $A$  of size  $n$ . In other words,  $\mathcal{S}$  can be covered by  $n$  subfamilies, each of which extends to an  $A$ -definable type.*

*Proof.* Let  $\mathcal{S} \subseteq \text{Def}(M^m)$  be an  $A$ -definable family of sets having a finite tame transversal of size  $n$ . By Theorem 5.2.11 there exists  $n$  formulas  $\varphi_1(x, y_1, a_1), \dots, \varphi_n(x, y_n, a_n)$ , with parameters  $a_1, \dots, a_n$  respectively, such that, for every  $i$ , the family  $\{\varphi(M^m, y_i, a_i) : y_i \in M^{l(y_i)}\}$  is a downward directed and, moreover, for every  $S \in \mathcal{S}$ , there is some  $i$  and  $b_i \in M^{l(y_i)}$  such that  $\varphi_i(M^m, b_i, a_i) \subseteq S$ .

Let  $X$  denote the set of points  $\langle c_1, \dots, c_n \rangle$  in  $\prod_i M^{l(a_i)}$  satisfying that the formulas  $\varphi_1(x, y_1, c_1), \dots, \varphi_n(x, y_n, c_n)$  have the properties described above for  $c_i = a_i$ . That is, each  $\varphi_i(x, y_i, c_i)$  describes a downward directed family and, for every  $S \in \mathcal{S}$ , there is one such family that is finer than  $S$ . This set is nonempty, since  $\langle a_1, \dots, a_n \rangle \in X$ . Note that  $X$  is  $A$ -definable.

By Proposition 5.2.15 (applied to a family with only one set) let  $p$  be an  $A$ -definable type extending  $X$  and  $\langle \mathbf{c}_1, \dots, \mathbf{c}_n \rangle$  a realization of  $p$  in  $\prod_i U^{l(a_i)}$ . Note in particular that, for every  $i$ ,  $\text{tp}(\mathbf{c}_i/M)$  is  $A$ -definable. By Proposition 5.2.15 let  $p_i \in S_m(U)$  be a type  $\mathbf{c}_i$ -definable type extending  $\{\varphi_i(x, y_i, \mathbf{c}_i) : y_i \in U^{l(y_i)}\}$ . Note that, by definition of  $X$ , every set  $S \in \mathcal{S}$  belongs in a type  $p_i$  for some  $i$ .

Finally for any  $i$  let  $q_i$  be the restriction of  $p_i$  to  $M$ . By Lemma 5.3.17 these restrictions are  $A$ -definable. Moreover every  $S \in \mathcal{S}$  belongs in  $q_i$  for some  $i$ .  $\square$

Note that, by a first-order logic compactness argument, a formula  $\varphi(x, b)$  does not  $k$ -divide over a small set  $A$  if and only if there exists  $\Omega \in \text{tp}(b/A)$  and some  $m \geq k$  such that  $\{\varphi(U^{l(x)}, u) : u \in \Omega\}$  has the  $(m, k)$ -property. Using this fact and Proposition 5.3.18 one may check that the next theorem is equivalent to a weaker form of Theorem 5.3.9 with  $k$  in place of  $n$  in its statement.

**Theorem 5.3.19.** *Let  $\mathcal{M}$  be o-minimal. Let  $\varphi(x, b)$  be a formula,  $l(x) = n$ , and  $A$  be a small set. The following are equivalent.*

(1)  $\varphi(x, b)$  does not  $(n + 1)$ -divide over  $A$ .

(2)  $\varphi(x, b)$  extends to an  $A$ -definable type.

Clearly a formula that extends to an  $A$ -definable type does not divide over  $A$ , and if it does not divide over  $A$  then in particular it does not  $k$ -divide over  $A$  for any  $k$ , so the two statements above are also equivalent to  $\varphi(x, b)$  not dividing over  $A$ . Moreover Cotter-Starchenko [9] (Corollary 5.6) showed that, in unpackable VC-minimal theories (in particular o-minimal theories), dividing and forking over sets are equivalent notions. So Theorem 5.3.19 implies in particular a parameter version of the o-minimal part of Theorem 5.3.16.

Finally, it is probably worth noting that the Alon-Kleitman-Matoušek  $(p, q)$ -theorem (Theorem 5.1.5) implies the equivalence between a formula  $\varphi(x, b)$  dividing over  $A$  and  $k$ -dividing over  $A$ , where  $k$  is any integer greater than the VC-codensity of  $\{\varphi(U^{l(x)} : b') : \text{tp}(b'/A) = \text{tp}(b/A)\}$ . By Theorem 5.1.6, in the o-minimal setting these two properties are also equivalent to having that  $\varphi(x, b)$   $(l(x) + 1)$ -divides over  $A$ .

## 5.4 Definable compactness

In this section we introduce several first order properties that aim to capture the notion of topological compactness among definable topological spaces in o-minimal structures. Relying on results from previous sections, we characterize definable compactness by proving the equivalence of these various notions (Theorem 5.4.9). We also prove the equivalence between definable compactness and compactness in o-minimal expansions of  $(\mathbb{R}, <)$  (Corollary 5.4.14). We include a preliminary subsection with the relevant definitions and some toolbox facts.

### 5.4.1 Definitions and basic facts

The following is our main definition.

**Definition 5.4.1.** *A definable topological space  $(X, \tau)$  is definably compact if every definable downward directed family of  $\tau$ -closed subsets of  $X$  has nonempty intersection.*

This notion has been explored in recent years by Johnson [27], Fornasiero [23] and the author, Thomas and Walsberg [2]. It differs from the classical definition of definable compactness within o-minimality [37], which we now present.

Let  $(X, \tau)$  be a definable topological space. Generalizing terminology from [37], we say that a curve  $\gamma : (a, b) \rightarrow X$  is  $\tau$ -*completable* if both  $\tau\text{-}\lim_{t \rightarrow a} \gamma$  and  $\tau\text{-}\lim_{t \rightarrow b} \gamma$  exist.

**Definition 5.4.2.** *A definable topological space  $(X, \tau)$  is definably curve-compact if every definable curve in  $X$  is  $\tau$ -completable.*

Definition 5.4.2 was introduced, with the name “definable compactness”, by Peterzil and Steinhorn [37], in the context of “definable spaces” in the sense of van den Dries [17] (Chapter 10). It has been used in more general settings by Thomas [48] and Walsberg [49]. With the aim of establishing a parallelism with general topology, we rename this property definable curve-compactness, and reserve the former label for a definition (Definition 5.4.1) in terms of families of closed sets.

The next definition is due to Fornasiero [23].

**Definition 5.4.3.** *Let  $(X, \tau)$  be a definable topological space and  $p, q$  be two types on  $X$ . The type  $q$  is a specialization of  $p$  if, for every  $\tau$ -closed set  $C \in p$ , it holds that  $C \in q$ . A point  $x \in X$  is a specialization of  $p$  if  $\text{tp}(x/M)$  is, i.e. if  $x$  is contained in every  $\tau$ -closed set in  $p$ .*

*We say that  $(X, \tau)$  is specialization-compact if, for every definable type  $p \in S_X(M)$ , there exists  $x \in X$  that is a specialization of  $p$ .*

Let  $(X, \tau)$  be a definable topological space and let  $\mathcal{N}$  be an elementary extension of  $\mathcal{M}$ . For any definable set  $S$ , let  $S(\mathcal{N})$  denote the interpretation of  $S$  in  $\mathcal{N}$ . Given  $x, y \in X(\mathcal{N})$ , it would be reasonable to describe  $y$  as being *infinitely close to  $x$  (with respect to  $\mathcal{M}$ )* if, for every definable (over  $M$ )  $\tau$ -neighborhood  $A$  of  $x$ , it holds that  $y \in A(\mathcal{N})$ . By the Marker-Steinhorn Theorem (Theorem 2.1.8), Definition 5.4.3 can then be interpreted as the condition that, in any tame extension  $\mathcal{N}$  of  $\mathcal{M}$ , every point in  $X(\mathcal{N})$  is infinitely close to a point in  $X$ .

**Remark 5.4.4.** In the paper by Thomas, Walsberg and the author [2] on which Chapter 4 is based we use Theorem 4.2.2 to prove the equivalence between definable compactness

and definable curve-compactness in o-minimal expansions of ordered groups ([2], Corollary 44), and Theorem 4.4.2 to prove the equivalence between definable curve-compactness and specialization-compactness in o-minimal expansions of ordered fields ([2], Corollary 47). We derive the equivalence between definable curve-compactness and classical compactness in o-minimal expansions of the field of reals ([2], Corollary 48). Since all these results are generalized in the next section we omit them in this thesis.

In the next proposition we present some basic facts around definable compactness. They correspond to Lemmas 3.5 (1), 3.8, 3.4 and 3.11 in [27]. The proofs follow mostly the treatment of the analogous results in general topology.

**Proposition 5.4.5.** *Let  $(X, \tau)$  and  $(Y, \mu)$  be a definable topological spaces.*

- (1) *If  $(X, \tau)$  is definably compact then, for any definable  $\tau$ -closed subset  $C$ , the subspace  $(C, \tau)$  is definably compact.*
- (2) *Suppose that  $(X, \tau)$  is Hausdorff and let  $C \subseteq X$  be a definable set such that the subspace  $(C, \tau)$  is definably compact, then  $C$  is  $\tau$ -closed.*
- (3) *Suppose that  $(X, \tau)$  is definably compact and let  $f : (X, \tau) \rightarrow (Y, \mu)$  be a definable continuous function. Its image  $(f(X), \mu)$  is definably compact.*
- (4) *Suppose that  $(X, \tau)$  is definably compact and  $(Y, \tau)$  is Hausdorff. If  $f : (X, \tau) \rightarrow (Y, \tau)$  is a definable continuous bijection then it is a homeomorphism.*

Peterzil and Steinhorn proved ([37], Theorem 2.1) that a euclidean space is definably curve-compact if and only if it is closed and bounded. Johnson proved the analogous fact for definable compactness ([27], Proposition 3.10). From Johnson's result one easily derives, using Proposition 5.4.5(3), the following lemma.

**Lemma 5.4.6.** *Let  $(X, \tau)$  be a definably compact definable topological space. Let  $f : (X, \tau) \rightarrow (M, \tau_e)$  be a definable continuous function. Then  $f$  reaches its maximum and minimum.*

Recall that every compact Hausdorff topological space is regular (in fact it is normal). We show that the analogous holds in our setting.

**Lemma 5.4.7.** *Any definably compact Hausdorff definable topological space  $(X, \tau)$  is regular.*

*Proof.* Towards a contradiction suppose that  $(X, \tau)$  is not regular. Let  $x \in X$  and  $C \subseteq X$  be a  $\tau$ -closed subset with  $x \notin C$  such that  $cl_\tau A \cap C \neq \emptyset$  for every  $\tau$ -neighborhood  $A$  of  $x$ . By passing to a larger  $C$  if necessary we may assume that it is definable. Let  $\mathcal{B}(x)$  be a definable basis of  $\tau$ -neighborhoods of  $x$ . Note that the definable family of  $\tau$ -closed sets  $\mathcal{C} = \{cl_\tau(A) \cap C : A \in \mathcal{B}(x)\}$  is downward directed. By definable compactness let  $y \in \bigcap \mathcal{C}$ . Then  $y$  belongs in the  $\tau$ -closure of any  $\tau$ -neighborhood of  $x$ , but this contradicts that the space is Hausdorff.  $\square$

We now include an example that illustrates how the condition of downward directedness in Definition 5.4.1 cannot be relaxed to simply having the FIP.

**Example 5.4.8.** Suppose  $\mathcal{M}$  expands a non-archimedean ordered group. Let  $r$  denote an infinitesimal element in  $\mathcal{M}$  with respect to another element 1. Then

$$\mathcal{S} = \{[0, 1] \setminus (x - r, x + r) : x \in [0, 1]\}$$

is a definable family of closed sets with the FIP but with empty intersection.

## 5.4.2 Characterizing definable compactness

The following is our main result on definable compactness.

**Theorem 5.4.9.** *Let  $(X, \tau)$  be a definable topological space. The following are equivalent.*

- (1)  $(X, \tau)$  is definably compact.
- (2)  $(X, \tau)$  is specialization-compact.
- (3) Any definable family of  $\tau$ -closed sets that extends to a definable type has nonempty intersection.
- (4) Any definable family of  $\tau$ -closed sets with the FIP has a finite transversal in  $X$ .
- (5) Any definable family  $\mathcal{C}$  of  $\tau$ -closed sets with the  $(m, n)$ -property, where  $m \geq n > \max\{1, \dim \cup \mathcal{C}\}$ , has a finite transversal in  $X$ .

(6) Any definable family  $\mathcal{C}$  of  $\tau$ -closed sets with the  $(m, n)$ -property, where  $m \geq n$  and  $n$  is greater than the VC-codensity of  $\mathcal{C}$ , has a finite transversal in  $X$ .

Moreover all the above imply and, if  $\tau$  is Hausdorff or  $\mathcal{M}$  has definable choice, are equivalent to:

(7)  $(X, \tau)$  is definably curve-compact.

**Remark 5.4.10.**

- (i) In [35] (Theorem 2.1) Peterzil and Pillay extracted from [12] the following. Suppose that  $\mathcal{M}$  has definable choice (e.g. expands an ordered group). Let  $\mathcal{U}$  be a monster model and  $\phi(x, b)$  be a formula in  $\mathcal{L}(U)$  such that  $\phi(U^{l(x)}, b)$  is closed and bounded. If the family  $\{\phi(U^{l(x)}, b') : \text{tp}(b'/M) = \text{tp}(b/M)\}$  has the finite intersection property, then  $\phi(U^{l(x)}, b)$  has a point in  $M^{l(x)}$ . They derive from this (Corollary 2.2 (i) in [35]) that any definable family of closed and bounded (equivalently definably compact with respect to the euclidean topology) sets with the finite intersection property has a finite transversal. Our work generalizes these results in a number of ways: we drop assumptions on  $\mathcal{M}$  besides o-minimality, and consider any  $M$ -definable topology. We also weaken the intersection assumption (in their work they actually observe that it suffices being  $k$ -consistent for some  $k$  in terms of  $l(x)$  and  $l(b)$ ) without the use of VC or forking theory.
- (ii)  $(p, q)$ -theorems are closely related to so-called Fractional Helly theorems (see [32]), which built on the classical Helly theorem. In its infinite version this theorem states that any family of closed and bounded convex subsets of  $\mathbb{R}^n$  that is  $(n + 1)$ -consistent has nonempty intersection. With an eye towards definably extending Lipschitz maps, Aschenbrenner and Fischer [5] proved (Theorem B) a definable version of Helly's Theorem (i.e. for definable families) in definably complete expansions of real closed fields. Our Theorem 5.4.9 and the arguments in Section 3.2 in [5] allow a generalization of the o-minimal part of this definable Helly Theorem, by asking that the sets be definably compact and closed in some definable topology instead of closed and bounded. More-

over, by appropriately adapting Corollary 2.6 in [5], one may show that every definable family of convex subsets of  $M^n$  that is  $(n + 1)$ -consistent extends to a definable type.

**Remark 5.4.11.** In Chapter 6 (Proposition 6.2.4) we prove the equivalence of definable compactness and definable curve-compactness for all one-dimensional definable topological spaces.

**Remark 5.4.12.** Recall the notion of definable net  $\gamma : (\Omega, \preceq) \rightarrow (X, \tau)$  described in Chapter 4 (Section 4.5). A *subnet* (a Kelley subnet) of  $\gamma$  is a net of the form  $\gamma' = \gamma \circ \mu$  where  $\mu : (\Omega', \preceq') \rightarrow (\Omega, \preceq)$  is a downward cofinal map. We say that  $\gamma'$  is a definable subnet if all of  $(\Omega', \preceq')$ ,  $\mu$  and  $\gamma$  are definable.

Classically a topological space is compact if and only if every net in it has a converging subnet. If  $\mathcal{M}$  has definable choice then one may show that definable compactness is equivalent to the condition that every definable net has a converging definable subnet. The proof follows the proof of the classical result, using only definable choice (otherwise it does not require o-minimality). See the paper of Thomas, Walsberg and the author [2] (Corollary 44) for a proof in o-minimal expansions of ordered groups.

We divide Theorem 5.4.9 into a number of propositions, all proving how definable compactness of  $(X, \tau)$  relates to the other properties. The equivalence between (1) and (2) is given by Proposition 5.4.13. The equivalence between (1), (4), (5) and (6) is given by Proposition 5.4.15. All the proofs make use of results from previous sections. Finally the implication (1) $\Rightarrow$ (7), and reverse implication when  $\tau$  is Hausdorff or  $\mathcal{M}$  has definable choice, are given by Proposition 5.4.17.

**Proposition 5.4.13.** *Let  $(X, \tau)$  be a definable topological space. Then  $(X, \tau)$  is definably compact if and only if it is specialization-compact.*

*Proof.* Suppose that  $(X, \tau)$  is specialization-compact and let  $\mathcal{C}$  be a definable downward directed family of  $\tau$ -closed sets. By Lemma 5.2.6  $\mathcal{C}$  extends to a definable type  $p$ . Let  $x \in X$  be a specialization of  $p$ , then  $x \in C$  for every  $C \in \mathcal{C}$ . It follows that  $(X, \tau)$  is definably compact.

Conversely suppose that  $(X, \tau)$  is definably compact, and let  $p \in S_X(M)$  be a definable type. Let  $\mathcal{B}$  denote a definable basis for the topology  $\tau$ . By Lemma 5.2.9 let  $\mathcal{F} \subseteq p$  be a downward directed definable family of sets that is complete for  $\mathcal{B}$ . By definable compactness there exists  $x \in \bigcap \{cl_\tau(F) : F \in \mathcal{F}\}$ . We show that  $x$  is a specialization of  $p$ .

Let us fix a closed set  $C \in p$ . If  $x \notin C$  then there is  $A \in \mathcal{B}$  such that  $x \in A$  and  $A \cap C = \emptyset$ . Since  $x \in \bigcap \{cl_\tau(F) : F \in \mathcal{F}\}$  we have that  $A \cap F \neq \emptyset$  for every  $F \in \mathcal{F}$ . Since  $\mathcal{F}$  is complete for  $\mathcal{B}$  it follows that there must be some  $F \in \mathcal{F}$  with  $F \subseteq A$ . But then  $F \cap C = \emptyset$ , contradicting that both  $F$  and  $C$  belong in  $p$ .  $\square$

**Corollary 5.4.14.** *Suppose that  $\mathcal{M}$  extends  $(\mathbb{R}, <)$ , the linear order of reals, and let  $(X, \tau)$  be a definable topological space. Then  $(X, \tau)$  is definably compact if and only if it is compact.*

*Proof.* If  $(X, \tau)$  is compact then it is clearly definably compact. We prove the direct implication.

Suppose that  $(X, \tau)$  is definably compact and let  $\mathcal{C}$  be a family of closed sets with the FIP. Let  $\mathcal{B}$  be a definable basis for  $\tau$ . We show that  $\bigcap \mathcal{C} \neq \emptyset$ . Note that, by definability of the topology, for every  $C \in \mathcal{C}$  and  $x \in X \setminus C$  there exists a definable open set  $A(x, C) \in \mathcal{B}$  such that  $x \in A(x, C)$  and  $A(x, C) \cap C = \emptyset$ . Then  $\bigcap \mathcal{C} = \bigcap \{X \setminus A(x, C) : C \in \mathcal{C}, x \in X \setminus C\}$ . So we may assume that all the sets in  $\mathcal{C}$  are definable, i.e.  $\mathcal{C}$  is a partial type.

Let  $p$  be a type expanding  $\mathcal{C}$ . By the Marker-Steinhorn Theorem (Theorem 2.1.8)  $p$  is definable. By Proposition 5.4.13 let  $x \in X$  be a specialization of  $p$ . Then  $x \in \bigcap \mathcal{C}$ .  $\square$

**Proposition 5.4.15.** *Let  $(X, \tau)$  be a definable topological space. The following are equivalent.*

- (1) *The topology  $\tau$  is definably compact.*
- (2) *Any definable family of  $\tau$ -closed sets that extends to a definable type has nonempty intersection.*
- (3) *Any definable family of  $\tau$ -closed sets with the FIP has a finite transversal in  $X$ .*
- (4) *Any definable family  $\mathcal{C}$  of  $\tau$ -closed sets with the  $(m, n)$ -property, where  $m \geq n > \max\{1, \dim \bigcup \mathcal{C}\}$ , has a finite transversal in  $X$ .*



(5) Any definable family  $\mathcal{C}$  of  $\tau$ -closed sets with the  $(m, n)$ -property, where  $m \geq n$  and  $n$  is greater than the VC-codensity of  $\mathcal{C}$ , has a finite transversal in  $X$ .

*Proof.* We prove that definable compactness (1) is equivalent to each of the other statements. By Theorem 5.2.11 any definable downward directed family extends to a definable type (in particular it has the FIP). If it admits a finite transversal, then it clearly (Fact 5.2.4) has nonempty intersection. Hence, (2), (3), (4) and (5) imply (1). We show that (1) $\Rightarrow$ (2). The implications (1) $\Rightarrow$ (4) and (1) $\Rightarrow$ (5) then follow from Theorem 5.3.9 and Corollary 5.3.11 respectively. The implication (1) $\Rightarrow$ (3) follows because (4) $\Rightarrow$ (3) is obvious.

The implication (1) $\Rightarrow$ (2) follows easily from applying specialization-compactness as follows (in general the equivalence between specialization compactness and (2) holds regardless of o-minimality). Suppose that  $\tau$  is definably compact and let  $\mathcal{C}$  be a definable family of  $\tau$ -closed sets that extends to a definable type  $p$ . By Proposition 5.4.13 there exists  $x \in X$  a specialization of  $p$ . Clearly  $x \in \bigcup \mathcal{C}$ .  $\square$

Finally we prove the connection between definable compactness and definable curve-compactness stated in Theorem 5.4.9. That is, that definable compactness implies definable curve-compactness in general, and that both notions are equivalent when the topology is Hausdorff or when the underlying structure  $\mathcal{M}$  has definable choice. This is Proposition 5.4.17. We follow the proposition with an example of a non-Hausdorff topological space definable in the trivial structure  $(M, <)$  that is definably curve-compact but not definably compact (Example 5.4.20).

The next lemma allows us to apply definable choice in certain instances even when the underlying structure  $\mathcal{M}$  does not have the property.

**Lemma 5.4.16** (Definable choice in compact Hausdorff spaces). *Let  $C \subseteq M^m$  be a definable nonempty  $\tau$ -closed set in a definably curve-compact Hausdorff definable topological space  $(X, \tau)$ . Suppose that  $\tau$  and  $C$  are  $A$ -definable. Then there exists a point  $x \in C \cap \text{dcl}(A)$ , where  $\text{dcl}(A)$  denotes the definable closure of  $A$ .*

*Consequently, for any  $A$ -definable family  $\{\varphi(u, M^m) : u \in \Omega\}$  of nonempty subsets of  $X$ , there exists an  $A$ -definable selection function  $s : \Omega \rightarrow X$  such that  $s(u) \in \text{cl}_\tau(\varphi(u, M^m))$  for every  $u \in \Omega$ .*

*Proof.* We prove the first paragraph of the lemma. The uniform result is derived in the usual way by the use of first-order logic compactness.

For this proof we adopt the convention of the one point euclidean space  $M^0 = \{0\}$ . In particular any projection  $M^k \rightarrow M^0$  is simply the constant function 0 and any relation  $E \subseteq M^0 \times M^k$  is definable if and only if its restriction to  $M^k$  is.

Let  $C$  and  $\tau$  be as in the Lemma. Let  $0 \leq n \leq m$  be such that there exists an  $A$ -definable function  $f : D \subseteq M^n \rightarrow C$ , for  $D$  a nonempty set. If  $n$  can be chosen to be zero then the lemma follows. We prove that this is the case by backwards induction on  $n$ .

Note that  $n$  can always be chosen equal to  $m$  by letting  $f$  be the identity on  $C$ . Suppose that  $0 < n \leq m$ . For any  $x \in M^{n-1}$  let  $D_x$  denote the fiber  $\{t \in M : \langle x, t \rangle \in D\}$ . For each  $x \in \pi(D)$  let  $s_x = \sup D_x$ , and consider the  $A$ -definable set  $H = \{x \in \pi(D) : s_x \in D_x\}$ .

If  $H \neq \emptyset$  then let  $g$  be the map  $x \mapsto f(s_x) : H \rightarrow C$ . If  $H = \emptyset$  then let  $g$  be the map  $x \mapsto \tau\text{-}\lim_{t \rightarrow s_x^-} f(\langle x, t \rangle) : \pi(D) \rightarrow C$  which, by definable curve-compactness and Hausdorffness, is well defined. In both cases  $g$  is an  $A$ -definable nonempty partial function  $M^{n-1} \rightarrow C$ . □

Lemma 5.4.16 implies that, even when  $\mathcal{M}$  does not have definable choice, if an  $A$ -definable family of sets  $\mathcal{S}$  in an  $A$ -definable definably curve-compact Hausdorff topology has a finite transversal, then it also has one in  $dcl(A)$ . To prove this it suffices to note that, for any  $k \geq 1$ , the set of  $k$ -tuples of points corresponding to a transversal of  $\mathcal{S}$  is  $A$ -definable and closed in the product topology.

**Proposition 5.4.17.** *Let  $(X, \tau)$  be a definable topological space. If  $(X, \tau)$  is definably compact then it is definably curve-compact.*

*Suppose that either  $\tau$  is Hausdorff or  $\mathcal{M}$  has definable choice. Then  $(X, \tau)$  is definably compact if and only if it is definably curve-compact.*

Proposition 5.4.17 gives a partial answer to a question by Johnson, who asked in [27] (Question 4.14) whether definable compactness and definable curve-compactness are equivalent notions for “definable spaces” in the sense of van den Dries [17]. Recall (Remark 5.4.11) that in Chapter 6 (Proposition 6.2.4) we prove that definable compactness and definable curve-compactness are also equivalent for all one-dimensional definable topological spaces.

We prove the left to right direction through a short lemma.

**Lemma 5.4.18.** *Let  $(X, \tau)$  be a definably compact definable topological space. Then  $(X, \tau)$  is definably curve-compact.*

*Proof.* Let  $\gamma : (a, b) \rightarrow X$  be a definable curve in  $X$ . Consider the definable family of  $\tau$ -closed nested sets  $\mathcal{C}_\gamma = \{cl_\tau(\gamma[(a, t)]) : a < t < b\}$ . By definable compactness there exists  $x \in \cap \mathcal{C}_\gamma$ . Clearly  $x \in \tau\text{-}\lim_{t \rightarrow a^-} \gamma(t)$ . Similarly one shows that  $\gamma$   $\tau$ -converges as  $t \rightarrow b$ .  $\square$

We now prove a simpler case of the left to right implication.

**Lemma 5.4.19.** *Let  $(X, \tau)$  be a definable topological space. Suppose that either  $\tau$  is Hausdorff or  $\mathcal{M}$  has definable choice. Let  $\mathcal{C}$  be a nested definable family of  $\tau$ -closed nonempty subsets of  $X$ . If  $(X, \tau)$  is definably curve-compact then  $\cap \mathcal{C} \neq \emptyset$ .*

*Proof.* Let  $\mathcal{C} = \{\varphi(u, M^m) : u \in \Omega\}$ ,  $\Omega \subseteq M^n$ , and  $(X, \tau)$  be as in the lemma. We proceed by induction on  $n$ .

**Base case:**  $n = 1$ .

Consider the definable totally preorder  $\preceq$  in  $\Omega$  given by  $u \preceq v$  if and only if  $\varphi(u, M^m) \subseteq \varphi(v, M^m)$ . If  $\Omega$  has a minimum  $u$  with respect to  $\preceq$  then  $\varphi(u, M^n) \subseteq C$  for every  $C \in \mathcal{C}$  and the result is obvious. We suppose that  $(\Omega, \preceq)$  does not have a minimum and consider the definable nested family  $\{(-\infty, u)_{\preceq} : u \in \Omega\}$ . By the base case in the proof of Lemma 5.2.6 this family extends to a definable type with a basis of sets  $\{(a, t) : t > a\}$  for some  $a \in M \cup \{-\infty\}$  or  $\{(t, a) : t < a\}$  for some  $a \in M \cup \{+\infty\}$ . We consider the former case, being the proof for the latter analogous. This means that, for every  $u \in \Omega$ , there exists  $t(u) > a$  such that  $v \preceq u$  for every  $a < v < t(u)$ .

If  $\mathcal{M}$  has definable choice or if  $\tau$  is Hausdorff (Lemma 5.4.16) there exists a definable function  $f : \Omega \rightarrow \cup \mathcal{C}$  satisfying that  $f(u) \in \varphi(u, M^m)$  for every  $u \in \Omega$ . Recall that, for every  $u, v \in \Omega$ , if  $v \preceq u$  then  $\varphi(v, M^m) \subseteq \varphi(u, M^m)$ , and in particular  $f(v) \in \varphi(u, M^n)$ . It follows that, for every  $u \in \Omega$ , there exists  $t(u) > a$  such that  $f(v) \in \varphi(u, M^n)$  for every  $a < v < t(u)$ . Let  $b > a$  be such that  $(a, b) \subseteq \Omega$  and  $\gamma$  be the restriction of  $f$  to  $(a, b)$ . We conclude that, for every  $C \in \mathcal{C}$ ,  $\tau\text{-}\lim_{t \rightarrow a} \gamma(t) \in C$ .

**Inductive step:**  $n > 1$ .

For any  $x \in \pi(\Omega)$  let  $\mathcal{C}(x)$  denote the family  $\{\varphi(x, t, M^m) : t \in \Omega_x\}$ , and set  $C(x) := \cap \mathcal{C}(x)$ . By the base case the definable family of  $\tau$ -closed sets  $\mathcal{D} = \{C(x) : x \in \pi(\Omega)\}$  does not contain the empty set. Clearly  $\cap \mathcal{D} = \cap \mathcal{C}$ . We observe that the family  $\mathcal{D}$  is nested and the result follows from the induction hypothesis.

Given  $x, y \in \pi(\Omega)$ , if for every  $C \in \mathcal{C}(x)$  there is  $C' \in \mathcal{C}(y)$  with  $C' \subseteq C$  then  $\cap \mathcal{C}(y) \subseteq \cap \mathcal{C}(x)$ . Otherwise there is  $C \in \mathcal{C}(x)$  such that, for every  $C' \in \mathcal{C}(y)$ , it holds that  $C \subseteq C'$ , in which case  $\cap \mathcal{C}(x) \subseteq C \subseteq \cap \mathcal{C}(y)$ .  $\square$

We may now prove the proposition.

*Proof of Proposition 5.4.17.* By Lemma 5.4.18 we must only prove the right to left implication. Let  $(X, \tau)$ ,  $X \subseteq M^m$ , be a definably curve-compact Hausdorff definable topological space. Let  $\mathcal{C} \subseteq \mathcal{P}(X)$  be a definable downward directed family of subsets of  $(X, \tau)$ . We show that  $\cap \{cl_\tau(C) : C \in \mathcal{C}\} \neq \emptyset$ .

We proceed by induction on  $n = \min\{\dim C : C \in \mathcal{C}\}$ . By Lemma 5.2.6 after passing to a finer family if necessary we may assume that  $\mathcal{C}$  is a complete family of cells. For the length of this proof, given two partial  $M_{\pm\infty}$ -valued functions  $f$  and  $g$  let  $(f, g) = \{\langle x, t \rangle : x \in \text{dom}(f) \cap \text{dom}(g) \text{ and } f(x) < t < g(x)\}$ , relaxing thus the classical notation by allowing that  $f$  and  $g$  have different domains.

If  $n = 0$  there exists a finite set in  $\mathcal{C}$  and so (Fact 5.2.4)  $\cap \mathcal{C} \neq \emptyset$ . We now prove the case  $n = m > 0$ . Hence suppose that every  $C \in \mathcal{C}$  is an open cell  $C = (f_C, g_C)$ , for definable continuous functions  $f_C, g_C : \pi(C) \rightarrow M_{\pm\infty}$  with  $f_C < g_C$ . For any  $C \in \mathcal{C}$  let  $D(C) = \cap \{cl_\tau((f_{C'}, g_{C'})) : C' \in \mathcal{C}\}$ . We first show that these sets are nonempty.

Let us fix  $C = (f, g)$  and, for any  $x \in \pi(C)$ , let  $C^0(x) = \tau\text{-lim}_{t \rightarrow f(x)^+} \langle x, t \rangle$ . If  $\tau$  is Hausdorff then this point is unique, otherwise we use definable choice to pick one such point definably on  $x$ . The definable set  $C^0 = \{C^0(x) : x \in \pi(C)\}$  has dimension  $\dim(C) - 1 = n - 1$ . For any  $C' = (f', g') \in \mathcal{C}$ , the definable set  $\{x \in \pi(C) \cap \pi(C') : f(x) < g'(x)\}$  is nonempty, since otherwise we would have  $C \cap C' = \emptyset$ . It follows that  $C^0 \cap cl_\tau((f, g')) \neq \emptyset$ . Note that the definable family  $\{C^0 \cap cl_\tau((f, g_{C'})) : C' \in \mathcal{C}\}$  is downward directed. By inductive hypothesis there is a point that belongs in the  $\tau$ -closure of  $C^0 \cap cl_\tau((f, g_{C'}))$ , in particular in  $cl_\tau((f, g_{C'}))$ , for all  $C' \in \mathcal{C}$ . So  $D(C) \neq \emptyset$ .

Note that, for every  $C \in \mathcal{C}$ ,  $D(C) \subseteq cl_\tau(C)$ . We now note that the definable family of nonempty sets  $\{D(C) : C \in \mathcal{C}\}$  is nested. Then, by Lemma 5.4.19,  $\bigcap\{cl_\tau(D(C)) : C \in \mathcal{C}\} \neq \emptyset$ , and thus  $\bigcap\{cl_\tau(C) : C \in \mathcal{C}\} \neq \emptyset$ .

Let us fix  $C_1 = (f_1, g_1), C_2 = (f_2, g_2) \in \mathcal{C}$ . We may cover  $B = \pi(C_1) \cap \pi(C_2)$  by the definable sets

$$B_1 = \{x \in S : f_1(x) \leq f_2(x)\} \text{ and } B_2 = \{x \in S : f_1(x) > f_2(x)\}.$$

Since  $\mathcal{C}$  is complete there exists some  $i \in \{1, 2\}$  such that  $\pi(C) \subseteq B_i$  for some  $C \in \mathcal{C}$ . Without loss of generality suppose that  $i = 1$ , and let us fix  $C_3 \in \mathcal{C}$  with  $\pi(C_3) \subseteq B_1$ . For any  $C = (f, g) \in \mathcal{C}$  let  $C' = (f', g') \in \mathcal{C}$  be such that  $C' \subseteq C \cap C_3$ . Clearly  $(f_2, g') \subseteq (f_1, g') \subseteq (f_1, g)$ . It follows that  $D(C_2) \subseteq D(C_1)$ . This completes the proof of the case  $n = m > 0$ .

Finally we prove the case  $0 < n < m$  by adapting the arguments above. Fix  $\hat{C} \in \mathcal{C}$  with  $\dim \hat{C} = n < m$  and a projection  $\hat{\pi} : M^m \rightarrow M^n$  such that  $\hat{\pi}|_{\hat{C}} : \hat{C} \rightarrow \hat{\pi}(\hat{C})$  is a bijection. By passing to a finer subfamily if necessary we may assume that every set in  $\mathcal{C}$  is contained in  $\hat{C}$ . It follows that the definable downward directed family  $\hat{\pi}(\mathcal{C}) = \{\hat{\pi}(C) : C \in \mathcal{C}\}$  contains only open cells in  $M^n$ . Note moreover that this family is complete.

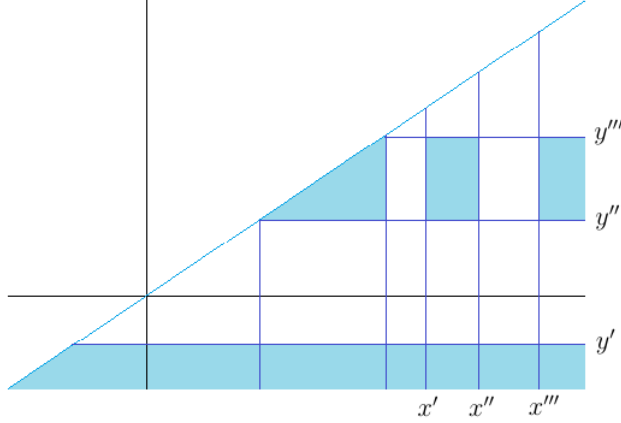
Set  $h := (\hat{\pi}|_{\hat{C}})^{-1}$ . Then the proof in the case  $n = m$  can be applied to  $\hat{\pi}(\mathcal{C})$  instead of  $\mathcal{C}$ , making sure to write throughout  $cl_\tau h(\cdot)$  in place of  $cl_\tau(\cdot)$ , and letting  $\hat{\pi}(C)^0 = \{\tau\text{-}\lim_{t \rightarrow f(x)^+} h(x, t) : x \in \pi(\hat{\pi}(C))\}$  for any  $\hat{\pi}(C) = (f, g) \in \hat{\pi}(\mathcal{C})$ . It follows that  $\bigcap\{cl_\tau(C) : C \in \mathcal{C}\} \neq \emptyset$ . This completes the proof of the proposition.  $\square$

The following is an example of a non-Hausdorff definable topological space in the o-minimal structure without definable choice  $(M, <)$  that is definably curve-compact but not definably compact.

**Example 5.4.20.** Let  $X = \{\langle x, y \rangle \in M^2 : y < x\}$ . Consider the family  $\mathcal{B}$  of sets

$$\begin{aligned} A(x', x'', x''', y', y'', y''') = & \{\langle x, y \rangle : y < y', y < x\} \cup \\ & \{\langle x, y \rangle : y'' < y < y''' \wedge (y < x < y''' \vee x' < x < x'' \vee x''' < x)\} \end{aligned}$$

definable uniformly over  $y' < y'' < y''' < x' < x'' < x'''$ .



**Figure 5.1.** Depicting (in blue) the set  $A(x', x'', x''', y', y'', y''')$ .

Given any  $A_0 = A(x'_0, x''_0, x'''_0, y'_0, y''_0, y'''_0)$  and  $A_1 = A(x'_1, x''_1, x'''_1, y'_1, y''_1, y'''_1)$  in  $\mathcal{B}$  and any  $\langle x, y \rangle \in A_1 \cap A_2$ , since every set in  $\mathcal{B}$  is open in the euclidean topology we may find  $y'' < y < y''' < x' < x < x''$  such that the box  $(x', x'') \times (y'', y''')$  is a subset of  $A_1 \cap A_2$ . Let  $y' < \min\{y'', y'_0, y'_1\}$  and  $x''' > \max\{x'', x''_0, x''_1\}$ . Then  $\langle x, y \rangle \in A(x', x'', x''', y', y'', y''') \subseteq A_1 \cap A_2$ . Hence the family  $\mathcal{B}$  is a definable basis for a topology  $\tilde{\tau}$ .

This topology is  $T_1$ , namely every singleton is closed. For every  $y \in M$ ,  $\tilde{\tau}\text{-}\lim_{t \rightarrow y^+} \langle t, y \rangle = \tilde{\tau}\text{-}\lim_{t \rightarrow +\infty} \langle t, y \rangle = (M \times \{y\}) \cap X$ , and, for every  $x \in M$ ,  $\tilde{\tau}\text{-}\lim_{t \rightarrow x^-} \langle x, t \rangle = (M \times \{x\}) \cap X$  and  $\tilde{\tau}\text{-}\lim_{t \rightarrow -\infty} \langle x, t \rangle = X$ . In particular  $\tilde{\tau}$  is not Hausdorff.

Now suppose that  $\mathcal{M} = (M, <)$ . By quantifier elimination we know that in this structure any definably partial map  $M \rightarrow M$  is piecewise either constant or the identity. Let  $\gamma = (\gamma_0, \gamma_1) : (a, b) \rightarrow X$  be an injective definable curve in  $X$ . Let  $I = (a, c) \subseteq (a, b)$  be an interval where  $\gamma_0$  and  $\gamma_1$  are either constant or the identity. Since the graph of the identity is disjoint from  $X$  and  $\gamma$  is injective it must be that  $\gamma_i$  is constant and  $\gamma_{1-i}$  is the identity on  $I$  for some  $i \in \{0, 1\}$ .

Suppose that  $i = 1$  with  $\gamma_1|_I$  having constant value  $y$ . Then, by the observations made above about the topology,  $\gamma$   $\tilde{\tau}$ -converges as  $t \rightarrow a$  to either  $\langle a, y \rangle$  (if  $y < a$ ) or  $(M \times \{y\}) \cap X$  (if  $a = y$ ). If  $i = 0$ , having  $\gamma_0|_I$  constant value  $x$ , then  $\gamma$   $\tilde{\tau}$ -converges as  $t \rightarrow a$  to either  $\langle x, a \rangle$  (if  $a < -\infty$ ) or the whole space  $X$  (if  $a = -\infty$ ). Treating the limit as  $t \rightarrow b$  similarly allows us to conclude that  $\gamma$  is  $\tilde{\tau}$ -completable. Hence the space  $(X, \tilde{\tau})$  is definably curve-compact.

On the other hand the definable downward directed family of nonempty  $\tilde{\tau}$ -closed sets  $\{X \cap M \times [u, +\infty) : u \in M\}$  has empty intersection. So  $(X, \tilde{\tau})$  is not definably compact.

We end the chapter with a question. Notice that all the characterizations of definable compactness in Theorem 5.4.9 are upfront expressible with infinitely many sentences in the language of  $\mathcal{M}$  (you have to check all relevant definable families of closed sets or all definable curves). In [37] it is shown that, in the euclidean topology, a set is definably curve-compact if and only if it is closed and bounded. Being closed and bounded is expressible by a single formula. Moreover, given a definably family of sets with the euclidean topology, the subfamily of those that are closed and bounded is definable. We ask the following.

**Question 5.4.21.**

- (i) *Can definable compactness of a given definable topology be expressed with a single formula in the language of  $\mathcal{M}$ ?*
- (ii) *More generally, given a definable family of definable topological spaces, is the subfamily of all which are definably compact definable?*

**5.A Proof of The Marker-Steinhorn Theorem**

In this appendix we use the approach of various proofs in this chapter, based on the the preorder described in Section 5.1.1, to extract from [31] a short proof of the Marker-Steinhorn Theorem (that avoids a treatment by cases). It is worth noting that a similar approach via definable linear orders was used by Walsberg [50] to prove the theorem in o-minimal expansions of ordered groups.

Recall that  $\mathcal{M}$  denotes an o-minimal structure and that, unless stated otherwise, definable means in  $\mathcal{M}$  over  $M$ . Recall that an elementary extension  $\mathcal{N}$  of  $\mathcal{M}$  is *tame* if, for every  $s \in N$ , the set  $\{t \in M : t < s\}$  has a supremum in  $M \cup \{+\infty\}$ . This is also referred to as  $\mathcal{M}$  being *Dedekind complete* in  $\mathcal{N}$ .

In full generality the Marker-Steinhorn Theorem (Theorem 2.1.8) shows that a type  $p \in S_n(M)$  is definable if and only if it is realized in some tame extension. We prove the “if” direction, and direct the reader to [31] for the proof of the reverse implication.

The approach of the proof below involves using cell decomposition to reduce the problem to showing that any cut in a definable preordered set that is realized in a tame extension is definable. This of course is obvious for the preordered set  $(M, \leq)$  by definition of tameness.

**Theorem 5.A.1.** *Let  $\mathcal{N} = (N, \dots)$  be a tame extension of  $\mathcal{M}$ . For every  $a \in N^n$ , the type  $\text{tp}(a/M)$  is definable.*

*Proof.* We must prove that, for every  $a \in N^n$  and formula  $\varphi(u, a)$ ,  $l(u) = m$ , the set  $\varphi(M^m, a) = \{u \in M^m : \mathcal{N} \models \varphi(u, a)\}$  is definable. We do this by induction on  $n$  and  $m$ , where in the inductive step we assume that it holds for any  $\langle n', m' \rangle$  smaller than  $\langle n, m \rangle$  in the lexicographic order. We may assume that  $a \notin M^n$ .

The case  $n = 1$  (for any  $m$ ) follows easily from tameness. In particular, let  $s_a$  be the supremum in  $M \cup \{\pm\infty\}$  of  $(-\infty, a) \cap M$ . If  $s_a < a$  then  $\text{tp}(a/M)$  has a definable basis of the form  $\{(s_a, t) : s_a < t \in M\}$ , otherwise it has a definable basis of the form  $\{(t, s_a) : s_a > t \in M\}$ .

Suppose that  $n > 1$  and let  $a = \langle c, d \rangle \in N^{n-1} \times N$ . Let  $\psi_i(N^m, N^n)$ ,  $0 \leq i \leq l$ , be a (0-definable) cell decomposition of  $\varphi(N^m, N^n)$ . For every  $u \in M^m$ , the set  $\varphi(u, N^n)$  is partitioned by the cells  $\psi_i(u, N^n)$ ,  $0 \leq i \leq l$ . In particular  $\mathcal{N} \models \varphi(u, a)$  if and only if  $\mathcal{N} \models \psi_i(u, a)$  for some  $i$ . So to prove the theorem it suffices to pass to an arbitrary  $0 \leq i \leq l$  and show that  $\psi_i(M^m, a)$  is definable. Hence without loss of generality we assume that all the sets  $\varphi(u, N^n)$  are cells.

By induction hypothesis  $\text{tp}(c/M)$  is definable. Suppose that there exists an  $M$ -definable partial function  $f : N^{n-1} \rightarrow N$  with  $f(c) = d$ . Then, for every  $u \in N^m$ , we have that  $\mathcal{N} \models \varphi(u, a)$  if and only if  $\mathcal{N} \models \exists t (\varphi(u, c, t) \wedge (f(c) = t))$ . And so the result follows.

Hence onwards we assume that there exists no  $M$ -definable partial function  $f$  with  $f(c) = d$ . In particular we have that every  $u \in M^m$  with  $\mathcal{N} \models \varphi(u, a)$  satisfies that  $\varphi(u, N^n)$  is a cell of the form  $(f_u, g_u)$  for two definable continuous functions  $f_u$  and  $g_u$ . Let  $\Omega = \{u \in M^m : \varphi(u, N^n) \text{ is of the form } (f_u, g_u) \text{ and } \mathcal{N} \models \exists t \varphi(u, c, t)\}$ . Clearly  $\varphi(M^m, a) \subseteq \Omega$ . Since  $\text{tp}(c/M)$  is definable the set  $\Omega$  is definable. We prove that the sets  $\{u \in \Omega : f_u(c) < d\}$  and  $\{u \in \Omega : g_u(c) > d\}$  are definable. Then their intersection equals  $\varphi(M^m, a)$ . The proof of definability is the same for both sets, so we show it only for the former.



Let  $\preceq$  be the total preorder on  $\Omega$  induced by  $\{f_u(c) : u \in \Omega\}$  described in Section 5.1.1, i.e. for  $u, w \in \Omega$ ,  $u \preceq w$  if and only if  $f_u(c) \leq f_w(c)$ . By definability of  $\text{tp}(c/M)$  the preordered set  $(\Omega, \preceq)$  is definable. Let  $P = \{u \in \Omega : f_u(c) < d\}$  and  $Q = \{u \in \Omega : f_u(c) > d\} = \Omega \setminus P$ . We must prove that  $P$  (equivalently  $Q$ ) is definable. If  $P$  has a supremum in  $\Omega \cup \{\pm\infty\}$  with respect to  $\preceq$  then the result is immediate, so we assume otherwise. In particular we have that, for every  $u \in P$  and  $v \in Q$ ,  $\dim(u, v)_{\preceq} > 0$ .

Note that, to prove the definability of  $P$ , it suffices to show that there exists a definable set  $P' \subseteq P$  that is cofinal in  $P$ , since then  $P = \{u \in \Omega : u \preceq v \text{ for some } v \in P'\}$ . Similarly it is enough to show the existence of a definable  $Q' \subseteq Q$  coinital in  $Q$ . So we may always pass to a subset  $\Omega' \subseteq \Omega$  such that either  $\Omega' \cap P$  is cofinal in  $P$  or  $\Omega' \cap Q$  is coinital in  $Q$ , and then prove definability of  $\Omega' \cap P$  or  $\Omega' \cap Q$ . Hence, by passing to a subset if necessary, we may assume that, for any  $u \in P$  and  $v \in Q$ ,

$$\dim(u, v)_{\preceq} = \dim \Omega. \quad (\star)$$

Moreover, note that we may also pass to an arbitrary set in any given definable finite partition of  $\Omega$ . In particular, by cell decomposition, we assume that  $\Omega$  is a cell.

Suppose that  $m = 1$ . For each  $u \in \Omega(\mathcal{N})$  with  $c \in \text{dom}(f_u)$  (note that this includes every  $u$  in  $\Omega$ ) let  $\hat{f}(u) = f_u(c)$ . Then  $\hat{f}$  is definable in  $\mathcal{N}$  over  $M \cup \{c\}$ . By o-minimality there exists a partition (definable over  $M \cup \{c\}$ ) of the domain of  $\hat{f}$  into points and intervals such that, on each interval,  $\hat{f}$  is continuous and either constant or strictly monotonic. Since  $\text{tp}(c/M)$  is definable the intersections of these cells with  $\Omega$  are definable. Note that, on any such intersection, the restriction of the preorder  $\preceq$  is either  $\leq$ ,  $\geq$ , or  $\leq \cup \geq$  (the trivial relation where any two points are indistinguishable), depending respectively on whether  $\hat{f}$  is strictly increasing, decreasing or constant. We fix one such interval  $I$  and show that  $I \cap P$  is definable.

If  $I \cap P = I \cap \Omega$  or  $I \cap Q = I \cap \Omega$  then the result is immediate. Otherwise there exist  $u, v \in I \cap \Omega$  such that  $\hat{f}(u) < d$  and  $\hat{f}(v) > d$ . By continuity there must exist  $r$  in the subinterval of  $I$  with endpoints  $u$  and  $v$  with  $\hat{f}(r) = d$ . By tameness  $J = (-\infty, d) \cap M$  is

definable. Finally note that  $\hat{f}|_I$  is not constant, and thus it is strictly monotonic. If it is increasing then it must be that  $J \cap I = P \cap I$  and otherwise  $J \cap I = Q \cap I$ .

Now suppose that  $m > 1$ . For every  $x$  in the projection  $\pi(\Omega)$ , let  $\preceq_x$  be the definable preorder on the fiber  $\Omega_x$  given by  $s \preceq_x t$  if and only if  $\langle x, s \rangle \preceq \langle x, t \rangle$ . Note that, following the arguments in the case  $m = 1$ , the sets  $P_x$  and  $Q_x$  are definable, and moreover  $\Omega_x$  can be partitioned into finitely many points and intervals where the restriction of  $\preceq_x$  is either  $\leq$ ,  $\geq$ , or  $\leq \cup \geq$ .

If there exists  $x \in \pi(\Omega)$  such that  $\{x\} \times P_x$  is cofinal in  $P$  or  $\{x\} \times Q_x$  is cointial in  $Q$ , then we are done. Suppose otherwise. We complete the proof by partitioning  $\Omega$  into finitely many definable sets with the following property. For each set  $\Sigma$  in the partition and  $x \in \pi(\Sigma)$ , either  $\Sigma_x \subseteq P_x$  or  $\Sigma_x \subseteq Q_x$ . Observe that the set  $\Theta$  of all  $x \in \pi(\Sigma)$  such that  $\Sigma_x \subseteq P_x$  is described by

$$x \in \pi(\Sigma), \forall t \in \Sigma_x, f_{(x,t)}(c) < d,$$

so, by induction hypothesis (applied in the case  $\langle n, m - 1 \rangle$ ), this set is definable. It follows that  $\Sigma \cap P = \bigcup_{x \in \Theta} (\{x\} \times \Sigma_x)$  is definable, and we may conclude that  $P$  is definable.

Recall that  $\Omega$  is a cell. If it is defined as the graph of a function then, by taking  $\Sigma$  in the above paragraph to be  $\Omega$ , we are done, so we assume otherwise. Let  $x \in \pi(\Omega)$ . By  $(\star)$ , since  $\{x\} \times P_x$  is not cofinal in  $P$  and  $\{x\} \times Q_x$  is not cointial in  $Q$ , the dimension of  $\{u \in \Omega : \{x\} \times P_x \prec u \prec \{x\} \times Q_x\}$  equals  $\dim \Omega$ . Consider  $r \in \Omega_x$  to be the right endpoint of some maximal subinterval  $I'$  of  $P_x$ . Then in particular  $r$  is the left endpoint of a subinterval  $I''$  of  $Q_x$ . If  $r \notin P_x$  then the set  $\{u \in \Omega : \{x\} \times I' \prec u \prec \langle x, r \rangle\}$  has dimension  $\dim(\Omega)$ . If however  $r \in P_x$  then the set  $\{u \in \Omega : \langle x, r \rangle \prec u \prec \{x\} \times I''\}$  has dimension  $\dim(\Omega)$ . If  $r$  is the right endpoint in  $\Omega_x$  of a maximal subinterval of  $Q_x$  then the analogous holds. Note that, if there exists  $s, t \in \Omega_x$  with  $s \in P_x$  and  $t \in Q_x$ , there will always be some  $r$  in the closed interval between  $s$  and  $t$  with the described properties.

For any  $u = \langle x, t \rangle \in \Omega$ , let  $L(x, t)$  be the set of  $v \in \Omega$  such that either  $\{x\} \times I' \prec v \prec u$  or  $u \prec v \prec \{x\} \times I'$  for some interval  $I' \subseteq \Omega_x$  with right endpoint  $t$ . Similarly let  $U(x, t)$  be the set of  $v \in \Omega$  such that either  $\{x\} \times I'' \prec v \prec u$  or  $u \prec v \prec \{x\} \times I''$  for some interval  $I'' \subseteq \Omega_x$  with left endpoint  $t$ . These sets are definable uniformly on  $u \in \Omega$ . Let  $\Lambda$

be the definable set of all  $u \in \Omega$  such that  $U(u) \cup L(u)$  has dimension  $\dim(\Omega)$ . By the above paragraph, for every  $x \in \pi(\Omega)$  and  $s, t \in \Omega_x$ , if  $s \in P_x$  and  $t \in Q_x$ , then there is some  $r$  in the closed interval between  $s$  and  $t$  such that  $\langle x, r \rangle \in \Lambda$ .

We now show that, for every  $x \in \Omega$ , the fiber  $\Lambda_x$  is finite. Then the proof is completed by taking any finite cell partition of  $\Omega$  compatible with  $\Lambda$ , since, for any  $\Sigma$  in said partition and  $x \in \pi(\Sigma)$ , the fiber  $\Sigma_x$  is going to be either a point or an interval contained in either  $P_x$  or  $Q_x$ .

Towards a contradiction suppose that  $\Lambda_x$  is infinite for some  $x \in \Omega$ . Let  $J'$  be a subinterval of  $\Lambda_x$  where  $\preccurlyeq_x$  is either  $\leq$ ,  $\geq$  or  $\leq \cup \geq$ . Then note that, for any distinct  $s, t \in J'$ , the sets  $L(x, s)$ ,  $U(x, s)$ ,  $L(x, t)$  and  $U(x, t)$  are pairwise disjoint. Since, for every  $t \in J'$ ,  $\dim(L(x, t) \cup U(x, s)) = \dim \Omega$ , we derive a contradiction from the Fiber Lemma for o-minimal dimension (Lemma 2.1.3).  $\square$

# 6. ONE-DIMENSIONAL DEFINABLE TOPOLOGICAL SPACES

## Introduction

In this chapter we study definable topological spaces  $(X, \tau)$  in o-minimal structures where  $\dim X \leq 1$ . We undertake their study in topological terms, considering different separation axioms and definable topological properties. We classify these spaces accordingly, and study different classes in terms of piecewise decompositions and existence of definable embeddings and homeomorphisms.

The axiom of o-minimality implies that the structure of one-dimension definable topological spaces is rather restrictive, admitting strong classification results, in particular when compared to spaces of higher dimensions. Nevertheless, among the examples we count definable versions of some rather classical topological spaces such as the Sorgenfrey Line, Split Interval and the Alexandrov Double Circle (Examples [A.3](#), [A.4](#) and [A.13](#) respectively). The thesis following from our results is that these few examples, albeit displaying a wide variety of distinct (definable) topological properties, act as building blocks describing large classes of one-dimensional definable topologies.

In [Section 6.1](#) we include some basic definitions, conventions and examples. [Section 6.2](#) contains preliminary results. In [Section 6.3](#) we prove some results on  $T_1$  and Hausdorff spaces. We show how Hausdorff one-dimensional definable topologies can be decomposed in terms of the euclidean, discrete and upper and lower limit topologies ([Theorem 6.3.9](#)). We observe that the Cantor Space is not a definable topological space ([Corollary 6.3.8](#)). We also prove the Gruenhage 3-element basis conjecture in our setting ([Remark 6.3.3](#)). In [Section 6.4](#) we prove a decomposition theorem for regular Hausdorff one-dimensional definable topologies ([Theorem 6.4.3](#)). In [Section 6.5](#) we show that the class of regular Hausdorff definable topologies in the line, as well as other classes of spaces, admit an almost definably universal space, in a sense that is made precise. We answer universality questions. In [Section 6.6](#) we prove that all regular Hausdorff definable topologies in the line can be Hausdorff compactified in a definable sense ([Theorem 6.6.6](#)). In [Section 6.7](#) we address the question of which one-dimensional definable topological spaces are affine in the setting of an o-minimal expansion

of an ordered field, proving in particular that it suffices for them to be piecewise euclidean (Theorem 6.7.1). In Section 6.8 we prove metrizability results. We prove a theorem implying that, in an o-minimal expansion of the field of reals, any one-dimensional definable topology that arises from a metric also admits a definable metric (Theorem 6.8.2). Section 6.9 describes the work of Peterzil and Rosel [36] on one-dimensional definable topologies in o-minimal structures, which was published during the writing of this thesis. We describe how their main result relates to some of ours, and answer some of their open questions.

Most of the results we present fail to generalize to spaces of higher dimensions, as noted in the examples in Appendix A.

This chapter is based on a paper in preparation with Margaret Thomas and Erik Walsberg.

## 6.1 Definitions and first examples

We fix infinitely many parameters  $0 < 1 < 2 < \dots$  in  $M$ , in such a way that it will be clear from context when these numerals denote elements of  $M$  and when they are just natural numbers. At times we assume that  $\mathcal{M}$  expands an ordered group  $(M, 0, +, <)$  or field  $(M, 0, 1, +, \cdot, <)$ , in which case these parameters have their natural interpretations. In Sections 6.7 and 6.8, where we assume throughout that our underlying structure expands an ordered field, we resort to notation  $\mathcal{R}$  and  $R$  in place of  $\mathcal{M}$  and  $M$  respectively.

See Chapter 2 (Section 2.2) for a review of basic notational conventions, definitions and results regarding definable topological spaces. We use almost all the content of that section in this chapter. Recall that a topological space is  $T_1$  if every singleton is closed,  $T_2$  if it is Hausdorff and  $T_3$  if it is Hausdorff and regular. Recall that, given a definable set  $X$ , we denote the euclidean and discrete topologies on  $X$  by  $\tau_e$  and  $\tau_s$  respectively, in such a way that the notation remains unambiguous.

Recall the notions of definable connectedness (Definition 2.2.9), definable separability (Definition 3.1.1), definable compactness (Definition 5.4.1), definable metrizability (Definition 2.2.6) and the frontier dimension inequality (f.d.i., Definition 2.2.10).

We present two definable topologies that are relevant to this chapter and which are immediate generalizations of classical topologies definable in  $(\mathbb{R}, <)$ . Since there will be no ambiguity we use some standard terminology to refer to them as understood in our setting.

The right half-open interval topology (or lower limit topology) on  $M$  (Example A.3), denoted  $\tau_r$ , is the topology with definable basis

$$[x, y) \text{ for } x, y \in M, x < y.$$

We reserve the name ‘‘Sorgenfrey Line’’ to refer to the classical space, namely  $(M, \tau_r)$  when  $M$  expands  $(\mathbb{R}, <)$ . Similarly the left half-open interval topology (or upper limit topology) on  $M$ , denoted  $\tau_l$ , is the topology with definable basis

$$(x, y] \text{ for } x, y \in M, x < y.$$

These spaces are clearly  $T_3$ . Much as in general topology, they work as counterexamples to a number of otherwise plausible sounding conjectures in our setting. We adopt all notational conventions with respect to  $\tau_r$  and  $\tau_l$  that were previously set for  $\tau_e$  and  $\tau_s$  (i.e. we implicitly address subspace topologies and write  $r$  and  $l$  in place of  $\tau_r$  and  $\tau_l$  respectively when used as subscripts or prefixes).

By o-minimality, for any definable set  $X \subseteq M^n$  the euclidean space  $(X, \tau_e)$  is definably separable. On the other hand, when  $X$  is infinite, the discrete space  $(X, \tau_s)$  is not. If  $n = 1$  then the spaces  $(X, \tau_r)$  and  $(X, \tau_l)$  are definably separable.

It is easy to note that, for a given infinite definable set  $X \subseteq M$ , the spaces  $(X, \tau_r)$ ,  $(X, \tau_l)$  and  $(X, \tau_s)$  are not definably compact, and the space  $(X, \tau_e)$  is definably compact if and only if  $X$  is  $e$ -closed and bounded (by [37], Theorem 2.1).

**Remark 6.1.1.** Clearly any order reversing bijection  $M \rightarrow M$  is a homeomorphism  $(M, \tau_r) \rightarrow (M, \tau_l)$ . Let  $\tau_*$  denote either  $\tau_r$  or  $\tau_l$ . Then for any distinct pair  $\tau, \mu \in \{\tau_e, \tau_*, \tau_s\}$  and intervals  $I, J \subseteq X$ , there is no definable homeomorphism  $(I, \tau) \rightarrow (J, \mu)$ .

This is obvious if one of the topologies is discrete. If  $\{\tau, \mu\} = \{\tau_e, \tau_*\}$  then it follows from the fact that, while  $(I, \tau_e)$  is definably connected,  $(J, \tau_*)$  is totally definably disconnected (i.e. every definably connected subspace is trivial).

In Lemma 2.2.8 we proved that the space  $(M, \tau_r)$  is not definably metrizable. The same proof applies if we consider any infinite definable set  $X \subseteq M$  in place of  $M$  and also if we put  $\tau_l$  in place of  $\tau_r$ . On the other hand both the euclidean and discrete topologies are definably metrizable on any definable set (with the implicit assumption that  $\mathcal{M}$  expands an ordered group).

Consider finitely many definable topological spaces  $(X_i, \tau_i)$ ,  $X_i \subseteq M^n$ ,  $0 \leq i \leq k$ , each having a basis  $\mathcal{B}_i$ . Their disjoint union corresponds to the definable set  $\bigcup_{0 \leq i \leq k} \{i\} \times X_i$  with definable topology given by basis  $\bigcup_{0 \leq i \leq k} \{i\} \times \mathcal{B}_i$ . Note that, for each  $i$ , the map  $X_i \rightarrow \bigcup_i \{i\} \times X_i : x \mapsto \langle i, x \rangle$  is a definable open embedding.

Given a definable function  $f : X \rightarrow Y$ , in this chapter we sometimes say that  $f$  is *e-continuous* or an *e-homeomorphism* if, as a map  $(X, \tau_e) \rightarrow (Y, \tau_e)$ ,  $f$  is respectively continuous or a homeomorphism.

## 6.2 Basic results

We present preliminary results about definable topological spaces  $(X, \tau)$  where  $\dim X \leq 1$  or more specifically when  $X \subseteq M$ , which we informally refer to as spaces “in the line”.

In the terminology of Chapter 4, the next lemma can be edited to show that every one-dimensional definable downward (respectively upward) directed set  $(\Omega, \preceq)$  admits a definable downward (respectively upward) cofinal map  $\gamma : ((a, b), \leq') \rightarrow (\Omega, \preceq)$ , regardless of whether  $\mathcal{M}$  expands an ordered field, where  $\leq'$  is either  $\leq$  or the dual order  $\geq$ .

**Lemma 6.2.1.** *Let  $\mathcal{S} \subseteq \mathcal{P}(M^n)$  be a definable downward directed family. Suppose there exists  $S \in \mathcal{S}$  with  $\dim S \leq 1$ . Then there exists a definable curve  $\gamma : (a, b) \rightarrow \bigcup \mathcal{S}$  and some  $c \in \{a, b\}$  such that, for every  $S \in \mathcal{S}$ ,  $\gamma(t) \in S$  for all  $a < t < b$  close enough to  $c$ .*

*Proof.* Since otherwise the result is obvious we assume  $\bigcap \mathcal{S} = \emptyset$ , in particular it must be that every set in  $\mathcal{S}$  is infinite. Since one may always restrict the domain of  $\gamma$  appropriately, we

construct  $\gamma$  without caring that its image is contained in  $\cup \mathcal{S}$ . We first prove the case where  $n = 1$ , showing that it suffices to let  $\gamma$  be the identity on some appropriate interval.

Suppose that  $n = 1$ . By the proof of the base case in Lemma 5.2.6, there exists a definable nested family of intervals  $\mathcal{C}$ , whose form is either  $\mathcal{C} = \{(a, t) : t > a\}$  for some  $a \in M \cup \{-\infty\}$ , or  $\mathcal{C} = \{(t, b) : t < b\}$  for some  $b \in M \cup \{\infty\}$ , that is finer than  $\mathcal{S}$ . In the former case it suffices to let  $\gamma$  to be the identity on  $(a, \infty)$ , and in the latter the identity on  $(-\infty, b)$ .

Now suppose that  $n > 1$ . Let us fix some  $\mathbf{S} \in \mathcal{S}$  and a cell decomposition of  $\mathcal{D}$  of  $\mathbf{S}$ . Let  $D \in \mathcal{D}$  be a cell satisfying that, for every  $S \in \mathcal{S}$ ,  $S \cap D \neq \emptyset$  (see Fact 5.2.4). Let  $f : J \rightarrow D$  be a definable homeomorphism from an interval onto  $D$ . Then apply the proof of the case  $n = 1$  to the family  $\{f^{-1}(S \cap D) : S \in \mathcal{S}\}$  to find an interval  $I$  and a endpoint  $c$  of  $I$  such that, for every  $S \in \mathcal{S}$ ,  $t \in f^{-1}(D \cap S)$  for all  $t \in I$  close enough to  $c$ . Finally, it suffices to take  $\gamma = f|_{J \cap I}$ .  $\square$

We derive two important consequences of Lemma 6.2.1, given by Lemma 6.2.2 and Proposition 6.2.4 below.

**Lemma 6.2.2** (Definable curve selection). *Let  $(X, \tau)$ ,  $\dim X \leq 1$ , be a definable topological space. Then  $(X, \tau)$  has definable curve selection. That is, if  $Y \subseteq X$  is a definable set and  $x \in X$ , then  $x$  belongs to the closure of  $Y$  if and only if there exists a definable curve  $\gamma$  in  $Y$  that converges to  $x$ .*

*Proof.* It follows readily from the definition of curve convergence that, if there exists a curve in  $Y$   $\tau$ -converging to  $x \in X$ , then  $x \in cl_\tau(Y)$ . We prove the converse.

Let  $(X, \tau)$  be a definable topological space. If  $X$  is finite the result is trivial so we assume that  $\dim X = 1$ . Let us fix  $x \in X$  and a definable set  $Y \subseteq X$  such that  $x \in cl_\tau Y$ . Let  $\mathcal{B}(x)$  be a definable basis of  $\tau$ -neighborhoods of  $x$ . Note that the family  $\mathcal{B}(x) \cap Y = \{A \cap Y : A \in \mathcal{B}(x)\}$  is downward directed. Now let  $\gamma$  and  $c$  be as described in Lemma 6.2.1. Then  $\gamma$  is a definable curve in  $Y$  and clearly  $\tau\text{-}\lim_{t \rightarrow c} \gamma(t) = x$ .  $\square$

Note that the above lemma holds in any definable topological space if we add the condition that either  $\dim Y \leq 0$  or that  $\dim_x(X, \tau) \leq 1$  (i.e. the local dimension of  $x$  in  $(X, \tau)$  is at most one).



Applying Lemma 6.2.2 and Proposition 2.2.14, we conclude the following. We will use this corollary often in this chapter.

**Corollary 6.2.3.** *Let  $(X, \tau)$  and  $(Y, \mu)$  be definable topological spaces, where  $\dim X \leq 1$ . Let  $f : (X, \tau) \rightarrow (Y, \mu)$  be a definable map. Then  $f$  is continuous at  $x \in X$  if and only if, for every definable curve  $\gamma : (a, b) \rightarrow X$  and  $c \in \{a, b\}$ , if  $\tau\text{-}\lim_{t \rightarrow c} \gamma(t) = x$  then  $\mu\text{-}\lim_{t \rightarrow c} (f \circ \gamma)(t) = f(x)$ .*

We may also use Lemma 6.2.1 to prove the equivalence between definable compactness (Definition 5.4.1) and definable curve-compactness (Definition 5.4.2) among one-dimensional spaces.

**Proposition 6.2.4.** *Let  $(X, \tau)$  be a definable topological space with  $\dim X \leq 1$ . Then  $(X, \tau)$  is definably compact if and only if it is definably curve-compact.*

*Proof.* By Theorem 5.4.9 any definably compact space is definably curve-compact. We prove the converse. We may assume that  $\dim X = 1$ .

Let  $\mathcal{C}$  be a downward directed definable family of  $\tau$ -closed subsets of  $X$ . Applying Lemma 6.2.1 let  $\gamma : (a, b) \rightarrow \bigcup \mathcal{C}$  be a definable curve and  $c \in \{a, b\}$  be such that, for every  $C \in \mathcal{C}$ ,  $\gamma(t) \in C$  for all  $t$  close enough to  $c$ . By definable curve-compactness let  $x = \tau\text{-}\lim_{t \rightarrow c} \gamma(t)$ . Then clearly  $x \in \bigcap \mathcal{C}$ . We conclude that  $(X, \tau)$  is definably compact.  $\square$

By virtue of Proposition 6.2.4, we refer throughout this chapter to “definable curve-compactness” as simply “definable compactness”, with reference to both notions.

Definable curve selection also allows us to prove the following facts regarding  $e$ -accumulation sets. For completeness and in accordance with the focus of this chapter we only prove them for one-dimensional spaces. Nevertheless one may show, using Corollary 4.3.4, that (a) in Proposition 6.2.5 holds for spaces of all dimensions whenever  $\mathcal{M}$  expands an ordered field and, adapting Corollary 3.10 in [27], that (b) holds in general for spaces of all dimensions. In the next proposition the euclidean closure means with respect to the extended euclidean topology on  $M_{\pm\infty}^m$ .

**Proposition 6.2.5.** *Let  $(X, \tau)$ ,  $\dim X \leq 1$ , be a definable topological space.*

(a) For any  $x \in X$ ,  $y \in M_{\pm\infty}^m$ , it holds that  $y \in E_x$  if and only if there exists an injective definable curve in  $X$   $\tau$ -converging to  $x$  and  $e$ -converging to  $y$ .

(b) Let  $Y \subseteq X$  be a definable set and  $x \in \partial_\tau Y$ . If  $\tau$  is  $T_1$  then  $E_x \cap cl_e Y \neq \emptyset$ .

*Proof.* The left to right implication in (a) is immediate. For the right to left implication fix  $x \in X$  and  $y \in M_{\pm\infty}^m$ . Consider the definable topology  $\mu$  on  $X$  where every  $z \neq x$  is isolated and where a basis of neighborhoods of  $x$  is given by the family  $\{\{x\} \cup (A \cap B \setminus \{y\}) : x \in A \in \tau, y \in B \in \tau_e\}$ . Clearly  $\mu$  is Hausdorff and finer than  $\tau$ . Since  $y \in E_x$ , the sets  $(A \cap B \setminus \{y\})$ , where  $x \in A \in \tau$  and  $y \in B \in \tau_e$ , are nonempty. In particular  $x$  is in the  $\mu$ -closure of  $X \setminus \{x\}$ . Applying Lemma 6.2.2 there exists a necessarily injective definable curve  $\gamma$   $\mu$ -converging (and thus  $\tau$ -converging) to  $x$ . By construction  $\gamma$  must  $e$ -converge to  $y$ .

To prove (b) note that, if  $x \in \partial_\tau Y$ , then by Lemma 6.2.2 there is a definable curve in  $Y$   $\tau$ -converging to  $x$ . If the topology is  $T_1$  one such curve cannot be constant and so, by o-minimality, can be assumed to be injective. By o-minimality said curve  $e$ -converges in  $M_{\pm\infty}^m$  and the result then follows from the left to right implication of (a).  $\square$

We now turn to the notion of  $e$ -accumulation set for definable topological spaces in the line.

**Lemma 6.2.6.** *Let  $(X, \tau)$ ,  $X \subseteq M$ , be a definable topological space.*

(a) Given  $x, y \in X$ ,  $y \in E_x$  if and only if one of the following holds.

(i) For any  $\tau$ -neighborhood  $A$  of  $x$  there exists  $z > y$  such that  $(y, z) \subseteq A$ .

(ii) For any  $\tau$ -neighborhood  $A$  of  $x$  there exists  $z < y$  such that  $(z, y) \subseteq A$ .

(b) It follows from (a) that, if  $(X, \tau)$  is Hausdorff then, for any  $y \in M$ , there exist at most two points  $x_0, x_1 \in X$  such that  $y$  belongs in both  $E_{x_0}$  and  $E_{x_1}$  (i.e. for any distinct  $x_0, x_1, x_2 \in X$ ,  $E_{x_0} \cap E_{x_1} \cap E_{x_2} = \emptyset$ ).

*Proof.* If (i) fails then by o-minimality there exists a  $\tau$ -neighborhood  $A'$  of  $x$  and  $z' > y$  such that  $(y, z') \cap A' = \emptyset$ . Similarly if (ii) fails there is a  $\tau$ -neighborhood  $A''$  of  $x$  and  $z'' < y$  with  $(z'', y) \cap A'' = \emptyset$ . So  $A' \cap A''$  is a  $\tau$ -neighborhood of  $x$  such that  $(z'', z') \cap A' \cap A'' \subseteq \{y\}$ . This contradicts that  $y \in E_x$ . The rest of the lemma is immediate.  $\square$

Lemma 6.2.6 motivates the following definition.

**Definition 6.2.7.** Let  $(X, \tau)$ ,  $X \subseteq M$ , be a definable topological space. For  $x \in X$  we define the right  $e$ -accumulation set of  $x$ , denoted  $R_x \subseteq E_x$ , to be the set of points  $y \in M_{\pm\infty}$  satisfying that, for any  $\tau$ -neighborhood  $A$  of  $x$ , there exists  $z > y$  such that  $(y, z) \subseteq A$ . Namely if  $\{A_u : u \in \Omega_x\}$  is a definable basis of  $\tau$ -neighborhoods of  $x$  in  $(X, \tau)$ , then

$$R_x = \{y \in M_{\pm\infty} : \forall u \in \Omega_x, \exists z > y, (y, z) \subseteq A_u\}.$$

So the set  $R_x \setminus \{-\infty\}$  is definable. Similarly the left  $e$ -accumulation set of  $x$ , denoted  $L_x \subseteq E_x$ , is defined to be the set of points  $y \in M_{\pm\infty}$  satisfying that, for any  $\tau$ -neighborhood  $A$  of  $x$ , there exists  $z < y$  such that  $(z, y) \subseteq A$ .

The following proposition follows from the definition of right and left  $e$ -accumulation set and Lemma 6.2.6.

**Proposition 6.2.8.** Let  $(X, \tau)$  be a definable topological space with  $X \subseteq M$  and  $x \in X$ . Then

- (a) the relations  $\{\langle x, y \rangle : y \in R_x\} \subseteq X \times M_{\pm\infty}$  and  $\{\langle x, y \rangle : y \in L_x\} \subseteq X \times M_{\pm\infty}$  are definable;
- (b)  $E_x = R_x \cup L_x$ ;
- (c) if  $(X, \tau)$  is Hausdorff then, for any  $y \in X \setminus \{x\}$ ,  $R_x \cap R_y = \emptyset$  and  $L_x \cap L_y = \emptyset$ .

**Remark 6.2.9.** By Lemma 2.2.20 and Proposition 6.2.8 (b), if  $(X, \tau)$  is  $T_1$  then  $R_x$  and  $L_x$  are finite for every  $x \in X$ .

**Remark 6.2.10.** Let  $\gamma$  be a definable curve in  $X$   $e$ -converging to some  $y \in M_{\pm\infty}$ . Recall Remark 2.2.12. If  $\gamma$  is injective then we may assume that it lies in either  $(y, +\infty)$  or  $(-\infty, y)$ . In the former case we say that  $\gamma$   $e$ -converges to  $y$  from the right and in the latter that it does so from the left. Note that, if  $\gamma$   $e$ -converges to  $y$  from the right (respectively left) and  $x \in X$ , then  $\gamma$   $\tau$ -converges to  $x$  if and only if  $y \in R_x$  (respectively  $y \in L_x$ ).

It turns out that, if  $(X, \tau)$  is  $T_1$ , then, for any  $x \in X$ , the sets  $R_x$  and  $L_x$  characterize a definable basis of neighborhoods for  $x$ . We show this in the next lemma.

**Lemma 6.2.11.** *Let  $(X, \tau)$ ,  $X \subseteq M$ , be a definable  $T_1$  topological space. Let  $x \in X$ . By Remark 6.2.9 sets  $R_x$  and  $L_x$  are finite. Set  $R_x := \{y_1, \dots, y_n\}$  and  $L_x := \{z_1, \dots, z_m\}$ . Consider the family  $\mathcal{U}(x)$  of sets*

$$\{x\} \cup \bigcup_{1 \leq i \leq n} (y_i, y'_i) \cup \bigcup_{1 \leq j \leq m} (z'_j, z_j)$$

*definable uniformly over  $(y'_1, \dots, y'_n, z'_1, \dots, z'_m) \in M^{n+m}$ , where  $y_i < y'_i$  and  $z'_j < z_j$ . The definable family  $\{U \cap X : U \in \mathcal{U}(x)\}$  is a basis of neighborhoods of  $x$  in  $(X, \tau)$ .*

*In particular, in the case where  $\mathcal{M}$  expands an ordered group and  $x$  has a bounded  $\tau$ -neighborhood (implying  $E_x \cap \{-\infty, +\infty\} = \emptyset$ ), we may take  $\mathcal{U}(x)$  to be of the form*

$$U(x, \varepsilon) := \{x\} \cup \bigcup_{y \in R_x} (y, y + \varepsilon) \cup \bigcup_{y \in L_x} (y - \varepsilon, y)$$

for  $\varepsilon > 0$ .

*By passing to a subfamily if necessary, we may (and will) always assume that  $\mathcal{U}(x)$  is a family of subsets of  $X$ .*

*Proof.* Let  $\mathcal{U}(x)$  be as in the lemma. By definition of  $R_x$  and  $L_x$  it clearly holds that, for every  $\tau$ -neighborhood  $A$  of  $x$ , there exists  $U \in \mathcal{U}(x)$  such that  $U \subseteq A \subseteq X$ . It therefore remains to prove that all sets in  $\mathcal{U}(x)$  are  $\tau$ -neighborhoods of  $x$ .

Towards a contradiction, suppose that there exists  $U \in \mathcal{U}(x)$  that is not a  $\tau$ -neighborhood of  $x$ . So  $x \in \partial_\tau(X \setminus U)$ . By Lemma 6.2.2 there exists a (necessarily injective by  $T_1$ -ness) definable curve  $\gamma : I \rightarrow X \setminus U$  that  $\tau$ -converges to  $x$  and that by o-minimality must  $e$ -converge to some  $a \in M_{\pm\infty}$ . By Remark 6.2.10 if  $\gamma$   $e$ -converges from the right then  $a \in R_x$  and otherwise  $a \in L_x$ . Either way by construction of  $\mathcal{U}(x)$  it follows that  $\gamma$  maps into  $U$ , a contradiction.  $\square$

From the above lemma it follows that, if  $(X, \tau)$  is a  $T_1$  definable topological space with  $X \subseteq M$ , a point  $x \in X$  is  $\tau$ -isolated if and only if  $E_x = \emptyset$ , and the identity map  $(X, \tau) \rightarrow (X, \tau_e)$  is continuous at  $x \in X$  if and only if  $E_x \subseteq \{x\}$ .

The following Lemma will be fundamental in proofs in later sections.

**Lemma 6.2.12.** *Let  $(X, \tau)$ ,  $X \subseteq M$ , be a definable topological space. Let  $f : I \subseteq X \rightarrow M$  be a function on an interval  $I = (a, b)$ ,  $a, b \in M_{\pm\infty}$ , such that, for every  $x \in I$ ,  $f(x) \in E_x$ . Suppose that  $f$  is  $e$ -continuous and strictly increasing (respectively decreasing). We extend  $f$  to a function  $[a, b] \rightarrow M_{\pm\infty}$  by letting  $f(a) = \lim_{x \rightarrow a} f(x)$  and  $f(b) = \lim_{x \rightarrow b} f(x)$ . For all  $y \in X$ , we have that*

(a) *for any  $x \in [a, b)$ , if  $x \in R_y$  then  $f(x) \in R_y$  (respectively  $f(x) \in L_y$ );*

(b) *for any  $x \in (a, b]$ , if  $x \in L_y$  then  $f(x) \in L_y$  (respectively  $f(x) \in R_y$ ).*

*Under the additional assumption that  $\tau$  is regular the converse also holds. Namely*

(c) *for any  $x \in [a, b)$ , if  $f(x) \in R_y$  (respectively  $f(x) \in L_y$ ) then  $x \in R_y$ ;*

(d) *for any  $x \in (a, b]$ , if  $f(x) \in L_y$  (respectively  $f(x) \in R_y$ ) then  $x \in L_y$ .*

*Proof.* We prove (a). Let  $y \in X$  and suppose that  $f$  is strictly increasing. Suppose that  $x \in [a, b) \cap R_y$ . If  $f(x) \notin R_y$  then by  $o$ -minimality there is  $z > f(x)$  and a  $\tau$ -neighborhood  $A$  of  $y$  such that  $(f(x), z) \cap A = \emptyset$ . Since  $x \in R_y$ , there is  $x' > x$  in  $I$  such that  $(x, x') \subseteq A$ . Since  $f$  is  $e$ -continuous and strictly increasing there is  $x < x'' < x'$  such that  $f(x'') \in (f(x), z)$ . So  $A$  is a  $\tau$ -neighborhood of  $x''$  and  $f(x'') \notin cl_e(A)$ , which contradicts that  $f(x'') \in E_{x''}$ . Similarly one may prove (b) in the increasing case. The proof in the decreasing case is analogous.

For (c) and (d) we prove again only the case where  $f$  is strictly increasing. We present the proof of (c), the proof of (d) being similar. Suppose that  $f(x) \in R_y$ , for  $x \in [a, b)$ , and let  $A$  be a  $\tau$ -neighborhood of  $y$ . Then there is some  $z > f(x)$  such that  $(f(x), z) \subseteq A$ . Since  $f$  is  $e$ -continuous and strictly increasing there is  $x < x'$  such that  $f[(x, x')] \subseteq (f(x), z)$ . For any  $x'' \in (x, x')$ , since  $f(x'') \in E_{x''}$  and  $(f(x), z) \subseteq A$ , it follows that  $x'' \in cl_\tau(A)$ , i.e.  $(x, x') \subseteq cl_\tau(A)$ . So we have shown that, for every  $\tau$ -neighborhood  $A$  of  $y$ , there exists

$x' > x$  such that  $(x, x') \subseteq cl_\tau(A)$ . If  $x \notin R_y$  then there must exist some  $x'' > x$  and some  $\tau$ -neighborhood  $A'$  of  $y$  such that  $(x, x'') \cap A' = \emptyset$ . But then by regularity there is  $A'' \subseteq A'$ , another  $\tau$ -neighborhood of  $y$ , such that  $cl_\tau(A'') \subseteq A'$ , and in particular such that  $(x, x'') \cap cl_\tau(A'') = \emptyset$ , a contradiction by the above.  $\square$

Throughout this chapter will will often use Remark 2.2.16 to argue that, whenever  $\mathcal{M}$  expands an ordered field, any one-dimensional definable topological space can be assumed to be, up to definable homeomorphism, a space in the line.

In the case where  $\mathcal{M}$  is not assumed to expand an ordered field, cell decomposition and the observations made in Remark 6.2.13 below will suffice to generalize many of the results on spaces in the line to all one-dimensional spaces.

**Remark 6.2.13.** Let  $(X, \tau)$ ,  $X \subseteq M^n$ ,  $n > 1$ , be a definable topological space, let  $I = (a, b) \subseteq M$ ,  $a, b \in M_{\pm\infty}$ , be an interval and let  $f : I \rightarrow f(I) \subseteq X$  be an  $e$ -homeomorphism (which we extend continuously to a function  $[a, b] \rightarrow M_{\pm\infty}^n$ ). Consider the definable total order  $\prec$  on  $cl_e f(I) = f([a, b])$  given by identifying  $cl_e f(I)$  with  $[a, b]$  through  $f$ , i.e. for every pair  $x, y \in cl_e f(I)$ , set  $x \prec y$  if and only if  $f^{-1}(x) < f^{-1}(y)$ . Accordingly for any  $x \prec y$  in  $cl_e f(I)$  let  $(x, y)_\prec$  denote the corresponding interval with respect to  $\prec$ .

By means of this identification we may generalize the notion of right and left  $e$ -accumulation point to points in  $cl_e f(Y)$ . That is, if  $x \in X$  and  $y \in cl_e f(I)$ , then  $y \in R_x$  if and only if, for every  $\tau$ -neighborhood  $U$  of  $x$ , there is  $z \in f(I)$  such that  $y \prec z$  and  $(y, z)_\prec \subseteq U$ . Similarly  $y \in L_x$  if and only if, for every  $\tau$ -neighborhood  $U$  of  $x$ , there is  $z \in f(I)$  such that  $z \prec y$  and  $(x, y)_\prec \subseteq U$ . Proposition 6.2.8 (a) and (c) generalize to this setting.

Now suppose  $\dim X \leq 1$ . By o-minimal cell decomposition there is a finite definable partition  $\mathcal{X}$  of  $X$  into cells such that for every  $C \in \mathcal{X}$  there is a projection  $\pi_C : C \rightarrow I_C \subseteq M$  that is an  $e$ -homeomorphism onto a cell. By passing to a pushforward of  $(X, \tau)$  we may assume that, for every distinct pair  $C, C' \in \mathcal{X}$ , we have  $cl_e C \cap cl_e C' = \emptyset$ . Then, for any  $C \in \mathcal{X}$ , let  $\prec_C$  be the order on  $cl_e C$  given by identifying  $cl_e C$  with  $cl_e I_C$  through  $\pi_C$ . Now let  $\{n(C) < \omega : C \in \mathcal{X}\}$  be a numbering of the cells in  $\mathcal{X}$  and let  $\prec$  be definable linear order on  $cl_e X$  such that, for any  $x \in cl_e C$  and  $y \in cl_e C'$ , where  $C, C' \in \mathcal{X}$ , we have that  $x \prec y$  if

and only if  $n(C) < n(C')$  or  $n(C) = n(C')$  and  $x \prec_C y$ . That is,  $\prec$  is the linear order induced by the lexicographic order given the push-forward  $x \mapsto \{n(C)\} \times \pi_C(x)$  for  $x \in cl_e(C)$ .

Given this convention the space  $(X, \tau)$  behaves very much like a space in the line. The definitions of right and left  $e$ -accumulation set immediately generalize to points  $x \in X$ , by saying that  $y \in cl_e C$  belongs in  $R_x$  or  $L_x$  if it does with respect to  $\prec_C$ .

Under this correspondence the statements and proofs of Proposition 6.2.8 and Lemma 6.2.11 generalize to this setting. Moreover, suppose that, for any  $C, C' \in \mathcal{X}$  and partial function  $f : C \rightarrow C'$  defined on an interval  $(a, b)_{\prec_C}$ , we consider that  $f$  is increasing or decreasing to mean with respect to  $\prec_C$  and  $\prec_{C'}$ . Then Lemma 6.2.12 and its proof also generalize to  $(X, \tau)$ .

Note that under this construction the definitions of sets  $R_x$  and  $L_x$ , for any  $x \in X$ , are dependent of the choice of cell decomposition  $\mathcal{X}$  of  $X$ .

The remaining results in this section are formulated as usual for spaces in the line. By Remark 6.2.13 they can be generalized to all one-dimensional spaces.

**Proposition 6.2.14.** *Let  $(X, \tau)$ ,  $X \subseteq M$ , be a Hausdorff definable topological space that satisfies the frontier dimension inequality. Then  $(X, \tau)$  is regular.*

*Proof.* We prove that, for any  $x \in X$  and open neighborhood  $A$  of  $x$ , there exists an open neighborhood  $U \in \tau$  of  $x$  such that  $cl_\tau U \subseteq A$ .

Let  $x \in X$  and let  $A \in \tau$  be an open neighborhood of  $x$ . By passing to a subset of  $A$  if necessary we may assume that  $A$  is definable. By the frontier dimension inequality  $\partial_\tau A$  is finite. Since  $(X, \tau)$  is Hausdorff there exists, for every  $y \in \partial_\tau A$ , an open neighborhood  $A_y \in \tau$  of  $x$  such that  $y \notin cl_\tau A_y$ . Let  $U = \bigcap_{y \in \partial_\tau A} A_y \cap A$ . Then  $U$  is open and  $x \in U \subseteq cl_\tau U \subseteq A$ .  $\square$

In Example A.8 we describe a definable topological space in the line that is  $T_1$  and has the f.d.i. but fails to be regular, justifying the Hausdorffness assumption in the above proposition. In Example A.10 we construct a two-dimensional Hausdorff space with the f.d.i. that it not regular, showing that Proposition 6.2.14 does not immediately generalize to spaces of dimension greater than one.

**Definition 6.2.15.** Let  $(X, \tau)$  be a definable topological space. We say that  $(X, \tau)$  is definably normal if, given any pair of disjoint definable  $\tau$ -closed sets  $B, C \subseteq X$ , there exist definable disjoint open sets  $U, V \subseteq X$  such that  $B \subseteq U$  and  $C \subseteq V$ .

We say  $(X, \tau)$  definably completely normal if any definable subspace of  $(X, \tau)$  is definably normal.

**Proposition 6.2.16.** Let  $(X, \tau)$ ,  $X \subseteq M$ , be a definable topological space. If  $(X, \tau)$  is  $T_1$  and regular then it is definably completely normal.

*Proof.* We suppose that  $(X, \tau)$ ,  $X \subseteq M$ , is  $T_1$  and regular and prove that it is definably normal. Since being  $T_1$  and regular are hereditary properties we conclude that  $(X, \tau)$  is definably completely normal.

Let  $B, C \subseteq X$  be disjoint  $\tau$ -closed definable sets in  $(X, \tau)$ . To prove the proposition it suffices to show the existence of a definable  $\tau$ -neighborhood  $U$  of  $B$  such that the  $\tau$ -closure of  $U$  is disjoint from  $C$ . We proceed by constructing a suitable partition of  $B$  into two sets,  $B = B' \cup B''$ , where  $B''$  is finite. By regularity of  $(X, \tau)$  there clearly exists a definable  $\tau$ -neighborhood  $U''$  of  $B''$  such that  $cl_\tau(U'') \cap C = \emptyset$ . We show the existence of a definable  $\tau$ -neighborhood  $U'$  of  $B'$  such that  $cl_\tau(U') \cap C = \emptyset$ . The proof is then completed by taking  $U = U' \cup U''$ .

First note that, since  $(X, \tau)$  is  $T_1$  and regular, the space is also Hausdorff.

Set  $E_B := \cup_{x \in B} E_x$ . Let  $int_e(E_B)$  be the euclidean interior of  $E_B$  and set  $B' := \{x \in B : E_x \subseteq int_e(E_B)\}$ . By o-minimality  $E_B \setminus int_e(E_B)$  is finite and so, by Hausdorffness and Lemma 6.2.6 (b),  $B'' = B \setminus B'$  is also finite. By Lemma 6.2.11,  $B' \cup int_e(E_B)$  is a  $\tau$ -neighborhood of  $B'$ . Set  $U' := B' \cup int_e(E_B) \setminus C$ .

For any  $x \in B$ , note that  $E_x \cap C$  must be a subset of  $bd_e(C)$ , since otherwise, by Lemma 6.2.11, we would have that  $x \in cl_\tau C$ , which contradicts that  $C$  is  $\tau$ -closed and disjoint from  $B$ . Hence in particular  $int_e(E_B) \cap C$  is finite. So  $U' = B' \cup int_e(E_B) \setminus C$  is cofinite in  $B' \cup int_e(E_B)$ . Recall that  $B' \cup int_e(E_B)$  is a  $\tau$ -neighborhood of  $B'$ . Since  $(X, \tau)$  is  $T_1$ , it follows that  $U'$  is also a  $\tau$ -neighborhood of  $B'$ . We now show that  $cl_\tau(U') \cap C = \emptyset$ , which completes the proof.



Towards a contradiction suppose that some  $x \in C$  is in the closure of  $U'$ . Then  $x$  must be in the closure of  $\text{int}_e(E_B) \setminus C$ . Set  $E'_B := \text{int}_e(E_B) \setminus C$ . If there is some  $\tau$ -neighborhood  $A$  of  $x$  such that  $A \cap E'_B$  is finite then, since the space is  $T_1$ ,  $A \setminus E'_B$  is also a  $\tau$ -neighborhood of  $x$ , which contradicts that  $x$  is in the closure of  $E'_B$ . On the other hand, suppose that for every  $\tau$ -neighborhood  $A$  of  $x$  the intersection  $A \cap E'_B$  is infinite. Then for every such  $A$  there exists an interval  $I \subseteq A \cap E'_B$ . Using Lemma 6.2.11 note that, for any  $y \in B$ , if  $E_y \cap I \neq \emptyset$ , then  $y \in \text{cl}_\tau I$ . So, in this case, for every  $\tau$ -neighborhood  $A$  of  $x$ ,  $B \cap \text{cl}_\tau A \neq \emptyset$ , which contradicts that  $(X, \tau)$  is regular.  $\square$

### 6.3 $T_1$ and Hausdorff ( $T_2$ ) spaces. Decomposition in terms of the $\tau_e, \tau_c, \tau_r$ and $\tau_l$ topologies

This section focuses on the properties of  $T_1$  and Hausdorff spaces in the line. The main result is Theorem 6.3.9, which states that any Hausdorff definable topological space  $(X, \tau)$  in the line can be definably partitioned into finitely many subspaces each of which has one of the  $\tau_e, \tau_s, \tau_r$  or  $\tau_l$  topologies. This is a partial improvement on the next proposition, which shows that every infinite  $T_1$  definable topological space in the line contains an interval subspace with one of these topologies. In Remark 6.3.3 this result is addressed in the context of the 3-element basis conjecture of set-theoretic topology.

**Proposition 6.3.1.** *Let  $(X, \tau)$ ,  $X \subseteq M$ , be an infinite  $T_1$  definable topological space. Then there exists an interval  $J \subseteq X$  such that  $(J, \tau) = (J, \tau_\square)$ , where  $\square$  is one of  $e, r, l$  or  $s$ .*

Recall that every infinite definable set contains a one-dimensional definable subset. By o-minimal cell decomposition the above proposition generalizes as follows.

**Corollary 6.3.2.** *Every infinite  $T_1$  definable topological space has a subspace that is definably homeomorphic to an interval with either the euclidean, right half-open interval, left half-open interval, or discrete topology.*

*Proof of Proposition 6.3.1.* By Lemma 2.2.20 for each  $x \in X$  the set  $E_x$  is finite. Suppose it is the case that there exist infinitely many points  $x \in X$  satisfying  $(x, \infty) \cap E_x = \emptyset$ . In that case let  $I \subseteq X$  be a bounded interval containing only such points and fix  $C > I$ .

Otherwise let  $I' \subseteq X$  be an interval such that  $(x, \infty) \cap E_x \neq \emptyset$  for every  $x \in I$  and consider the definable map  $f$  on  $I$  taking each  $x$  to the smallest  $y > x$  such that  $y \in E_x$ . The map  $f$  satisfies  $x < f(x)$  for all  $x \in X$  and so by o-minimality, after passing to a subinterval where  $f$  is continuous and then applying continuity, there exist an interval  $I \subseteq I'$  and  $C > I$  such that, for all  $x \in I$ ,  $f(x) > C$ . In either case we have that, for all  $x \in I$ ,  $(x, C] \cap E_x = \emptyset$ . Similarly we can isolate a bounded subinterval  $J \subseteq I$  and some  $c < J$  such that, for every  $x \in J$ ,  $[c, x) \cap E_x = \emptyset$ . Thus we have reached an interval  $J$ ,  $c < J < C$ , such that for all  $x \in J$ ,  $E_x \cap cl_e J \subseteq \{x\}$ .

For any  $x \in J$  let  $\mathcal{U}(x)$  denote a family of neighborhoods of  $x$  as described in Lemma 6.2.11. Note that, by construction of  $J$ , for any given  $x \in J$  and  $y < x < z$  there is  $U \in \mathcal{U}(x)$  such that,

$$U \cap J = \begin{cases} (y, z) & \text{if } x \in R_x \cap L_x, \\ [x, z) & \text{if } x \in R_x \setminus L_x, \\ (y, x] & \text{if } x \in L_x \setminus R_x, \\ \{x\} & \text{if } x \notin R_x \cup L_x. \end{cases} \quad (6.1)$$

Recall that the families  $\{R_x : x \in J\}$  and  $\{L_x : x \in J\}$  are definable. Thus we may partition  $J$  into four definable sets  $J_i$ ,  $1 \leq i \leq 4$ , where

$$\begin{aligned} J_1 &= \{x \in J : x \in R_x \cap L_x\}, \\ J_2 &= \{x \in J : x \in R_x, x \notin L_x\}, \\ J_3 &= \{x \in J : x \notin R_x, x \in L_x\}, \\ J_4 &= \{x \in J : x \notin R_x \cup L_x\}. \end{aligned}$$

By (6.1) and definition of  $R_x$  and  $L_x$  the subspace topology on  $J_1$  is  $\tau_e$ . Similarly the subspace topologies on  $J_2$ ,  $J_3$  and  $J_4$  are  $\tau_r$ ,  $\tau_l$  and  $\tau_s$  respectively. At least one of the four

definable sets  $J_1$ ,  $J_2$ ,  $J_2$  and  $J_4$  must contain an interval, and so the result follows. At least one of these four sets must be infinite and thus contain an interval, the result follows.  $\square$

For a justification for the condition of  $T_1$ -ness in Proposition 6.3.1 see Example A.6, which describes a  $T_0$  definable topological space in the line that fails to be  $T_1$  and does not contain an interval with one of the  $\tau_e$ ,  $\tau_r$ ,  $\tau_l$  or  $\tau_s$  topologies.

**Remark 6.3.3.** Let  $(\star)$  denote the following condition: if  $(X, \tau)$  is an uncountable first countable regular Hausdorff topological space, then  $(X, \tau)$  contains a subset of the reals of cardinality  $\aleph_1$  with either the euclidean, Sorgenfrey or discrete topologies. The *3-element basis conjecture for uncountable first countable regular spaces* is an open conjecture in set-theoretic topology stating that ZFC plus the proper forcing axiom implies  $(\star)$ . It dates back to Gruenhage [25], and since then has been the object of ongoing research [34], [21]. It is known that the failure of  $(\star)$  is consistent with ZFC.

It is worth noting the relation between this conjecture and Corollary 6.3.2, which proves in particular the existence, in a definable sense, of a 3-element basis for the class of  $T_1$  infinite spaces definable in an o-minimal expansion of the field of reals.

We now make use of Proposition 6.3.1 to prove that the Cantor Space does not exist (up to homeomorphism) in the form of a definable topological space.

Recall the classical definition of the *weight* of a topological space  $(X, \tau)$ ,  $w_\tau(X)$ , namely the minimum cardinality of a basis for  $\tau$ .

**Remark 6.3.4.** Let  $(X, \tau)$ ,  $X \subseteq M$ , be a definable topological space. Clearly  $w_\tau(X) \leq |M|$ . We note that, if  $\tau \in \{\tau_r, \tau_l, \tau_s\}$ , then  $w_\tau(X) = |X|$ .

Clearly  $w_{\tau_s}(X) = |X|$ . We show that  $w_{\tau_r}(X) = |X|$ . Let  $\mathcal{B}$  be a basis of  $(X, \tau_r)$ . Then there must exist, for every  $x \in X$ , some  $A \in \mathcal{B}$  such that  $x \in A \subseteq [x, +\infty)$ . A map  $X \rightarrow \mathcal{B}$  that takes each  $x \in X$  to one such neighborhood  $A$  must be injective, so  $|X| \leq |\mathcal{B}| \leq w_\tau(X)$ . The other inequality is obvious. An analogous argument shows that  $w_{\tau_l}(X) = |X|$ .

From Proposition 6.3.1 it follows that, if  $(X, \tau)$  is  $T_1$  and infinite, then  $\alpha \leq w_\tau(X)$ , where  $\alpha = \min\{w_e(I) : I \subseteq X \text{ is an interval}\}$ .

Recall that, if  $\mathcal{M}$  expands a field, then any two intervals are definably  $e$ -homeomorphic, and so, if  $X$  is infinite,  $w_{\tau_s}(X) = w_{\tau_r}(X) = w_{\tau_l}(X) = |M|$ . Additionally, the  $\alpha$  defined above

equals  $w_e(M)$  and, since one may show that  $|M| \leq 2^{w_e(M)}$ , we have that  $w_e(M) \leq w_\tau(X) \leq 2^{w_e(M)}$ .

**Lemma 6.3.5.** *Let  $(X, \tau)$  be a  $T_1$  definable topological space, let  $I \subseteq M$  be an interval and let  $\gamma : I \rightarrow X$  be an injective definable curve. If  $(X, \tau)$  is compact then  $(I, <)$  is Dedekind complete (i.e. every nonempty subset of  $I$  that is bounded above in  $I$  admits a supremum).*

*Proof.* Towards a contradiction suppose that there exists a nonempty set  $S \subset I$  bounded above in  $I$  but with no supremum. Let  $S'$  be the set of upper bounds of  $S$  in  $I$ . Consider the following family of closed nonempty subsets of  $(X, \tau)$ :

$$\mathcal{S} = \{cl_\tau(\gamma[(t, s)]) : t \in S, s \in S'\}.$$

This family clearly has the finite intersection property. By compactness of  $(X, \tau)$  we may fix  $x \in X$  belonging in  $\cap \mathcal{S}$ . If  $x \notin \gamma(I)$  then we fix some  $t_x \in I$  and set  $\gamma_x := \gamma \setminus \{\{t_x, \gamma(t_x)\}\} \cup \{t_x, x\}$ , otherwise let  $\gamma_x = \gamma$ . Let  $(I, \tau_x)$  be the push-forward of  $(\gamma_x(I), \tau)$  by  $\gamma_x^{-1}$ .

Now note that, for every  $t, s \in I$  and every  $\tau_x$ -neighborhood  $U$  of  $t_x$ , if  $t \in S$  and  $s \in S'$  then  $U \cap (t, s) \neq \emptyset$ . In particular this holds for any  $t \in S$  with  $t > S \cap E_x^{(I, \tau_x)}$  and  $s \in S'$  with  $s < S' \cap E_x^{(I, \tau_x)}$ . This is clearly in contradiction with Lemma 6.2.11. So  $I$  is Dedekind complete.  $\square$

**Remark 6.3.6.** Since in an ordered field any two intervals are order-isomorphic, from Lemma 6.3.5 it follows that, under the assumption that  $\mathcal{M}$  expands an ordered field, if there exists an infinite compact  $T_1$  definable topological space then  $M$  is Dedekind complete. In particular, since  $\mathbb{R}$  is, up to (unique) field isomorphism, the only Dedekind complete ordered field, it must be that  $\mathcal{M}$  is an expansion of the field of reals.

**Proposition 6.3.7.** *There exists no infinite definable topological space  $(X, \tau)$  that is compact, totally disconnected, and that satisfies  $w_\tau(X) < |Y|$  for every  $Y \subseteq X$  that is infinite and definable.*

*Proof.* Let  $(X, \tau)$  be an infinite compact totally disconnected definable topological space satisfying  $w_\tau(X) < |Y|$  for every  $Y \subseteq X$  that is infinite and definable. We reach a contradiction by showing that  $(X, \tau)$  contains an infinite (and in fact definable) connected subspace.

First note that, since  $(X, \tau)$  is totally disconnected, in particular it is  $T_1$ . Let  $I \subseteq M$  be an interval and let  $\gamma : I \rightarrow X$  be an injective definable curve. Let  $(I, \tau_I)$  be the push-forward of  $(\gamma(I), \tau)$  by  $\gamma^{-1}$ . Since  $(I, \tau_I)$  is  $T_1$  then by Proposition 6.3.1 we may assume, after passing to a subinterval if necessary, that  $\tau_I \in \{\tau_e, \tau_r, \tau_l, \tau_s\}$ . Since by hypothesis  $w_{\tau_I}(I) = w_\tau(\gamma(I)) \leq w_\tau(X) < |f(I)| = |I|$  it must be that  $\tau_I = \tau_e$ . Now recall that, by compactness and Lemma 6.3.5,  $(I, <)$  is Dedekind complete, and so  $(I, \tau_e)$  is connected. Hence  $(\gamma(I), \tau)$  is connected.  $\square$

**Corollary 6.3.8.** *The Cantor space  $2^\omega$  is not a definable topological space.*

*Proof.* Let  $(X, \tau)$  be (homeomorphic to) the Cantor space and towards a contradiction suppose that it is a definable topological space. Recall that the Cantor space is compact, second countable (i.e.  $w_\tau(X) \leq \omega$ ) and totally disconnected. Let  $Y \subseteq X$  be an infinite definable set. We show that  $|Y| = \mathfrak{c}$ . Then the result follows from Proposition 6.3.7.

Clearly  $|Y| \leq \mathfrak{c}$ . By o-minimality there exists an interval  $I \subseteq M$  and an injective definable curve  $\gamma : I \rightarrow X$ . By Lemma 6.3.5  $(I, <)$  is Dedekind complete, so  $\mathfrak{c} \leq |I| = |\gamma(I)| \leq |Y|$ .  $\square$

It follows from the above corollary that the class of definable topological spaces up to homeomorphism is not closed under countable products.

If we impose Hausdorffness we can strengthen Proposition 6.3.1 as follows. See Example A.7 for a  $T_1$  non-Hausdorff space that cannot be decomposed as described below.

**Theorem 6.3.9.** *Let  $(X, \tau)$ ,  $X \subseteq M$ , be a nonempty Hausdorff definable topological space. Then there exists a finite partition  $\mathcal{X}$  of  $X$  into points and intervals such that, for every  $I \in \mathcal{X}$ ,  $\tau|_I \in \{\tau_e, \tau_r, \tau_l, \tau_s\}$ .*

*Proof.* We start by proving a simple case. Suppose that, for every  $x \in X$ ,  $E_x^{(X,\tau)} \subseteq \{x\}$ . We call this condition  $(\dagger)$ . Then let us partition  $X$  into four definable sets as follows.

$$\begin{aligned} & \{x \in X : x \in R_x \cap L_x\}, \\ & \{x \in X : x \in R_x, x \notin L_x\}, \\ & \{x \in X : x \notin R_x, x \in L_x\}, \\ & \{x \in X : x \notin R_x \cup L_x\}. \end{aligned}$$

By Lemma 6.2.11 these correspond respectively to spaces with the  $\tau_e, \tau_r, \tau_l$  and  $\tau_s$  topologies. By o-minimality we can partition each of these into a finite number of points and intervals, and the result follows.

In order to prove the theorem it is enough to show that we may partition  $(X, \tau)$  into finitely many definable subspaces where  $(\dagger)$  holds. We do so as follows.

Note that, for any definable subspace  $S \subseteq X$  and any  $x \in S$ ,  $E_x^{(S,\tau)} \subseteq E_x^{(X,\tau)}$ . We prove the existence of a finite partition of  $X$  formed by points and intervals such that, for any interval  $I$  in the partition and any  $x \in I$ ,  $E_x^{(X,\tau)} \cap cl_e I \subseteq \{x\}$ . Since any element in  $E_x^{(I,\tau)}$  must belong in  $cl_e I$  (Proposition 2.2.18 (a)) it follows that, for any  $x \in I$ ,  $E_x^{(I,\tau)} = E_x^{(X,\tau)} \cap cl_e I \subseteq E_x^{(X,\tau)} \cap cl_e I \subseteq \{x\}$ , i.e.  $(\dagger)$  holds in  $(I, \tau)$ , which completes the proof.

From now on for any  $x \in X$  let  $E_x = E_x^{(X,\tau)}$ . By Lemma 2.2.20, for any  $x \in X$  the set  $E_x$  is finite. By o-minimality (uniform finiteness) there exists some  $n$  such that  $|E_x| \leq n$  for every  $x \in X$ . We may partition  $X$  into finitely many definable subspaces  $X_0, \dots, X_n$ , where  $X_i = \{x \in X : |E_x| = i\}$  for  $0 \leq i \leq n$ . We fix  $Y = X_m$  for some  $0 \leq m \leq n$  and prove the existence of a partition of  $Y$  with the desired properties. Since otherwise the result is trivial we assume that  $m > 0$  and that  $Y$  is infinite.

For  $1 \leq i \leq m$ , let  $f_i : Y \rightarrow M_{\pm\infty}$  be the definable function taking each element in  $x \in Y$  to the  $i$ -th smallest element in  $E_x$ . Since the family  $\{E_x : x \in Y\}$  is definable these maps are definable. Moreover by Hausdorffness (see Lemma 6.2.6 (ii)) these functions cannot be constant on any interval. By o-minimality let  $\mathcal{Y}$  be a partition of  $Y$  into finitely many

intervals and points such that, for every interval  $I \in \mathcal{Y}$ , the functions  $f_i$ ,  $1 \leq i \leq m$ , are  $e$ -continuous and strictly monotone.

Without loss of generality we fix an interval  $I \in \mathcal{Y}$  and show that, for any  $x \in I$ ,  $E_x \cap cl_e I \subseteq \{x\}$ , completing the proof. Let  $x \in I$  and  $y \neq x$  be such that  $y \in E_x \cap cl_e I$ . If  $y \in I$  then, by Lemma 6.2.12,  $E_y \subseteq E_x$ . Since  $|E_y| = |E_x|$  it follows that  $E_y = E_x$ , contradicting that the functions  $f_i$  are injective. Suppose now that  $x \in \partial_e I$ . Then  $y = f_i(x)$  for some  $1 \leq i \leq m$ . By  $e$ -continuity and strict monotonicity of  $f_i$  on  $I$  there exists a point  $x' \in I$  such that  $f_i(x') \in I$  and  $f_i(x') \neq x'$ . A contradiction then follows as before.  $\square$

By o-minimal cell decomposition the above theorem has an immediate generalisation to all one-dimensional spaces. Note that it follows that any definable function  $(X, \tau) \rightarrow (M, \tau_e)$ , where  $\dim X \leq 1$  and  $\tau$  is a Hausdorff definable topology, is cell-wise continuous.

We end this section with a remark noting that, for spaces in the line, having an interval subspace with either the euclidean, discrete or half-open interval topologies is a definable topological invariant. It is an easy consequence of the monotonicity theorem of o-minimality. It holds in weakly o-minimal structures too, since these have a form of monotonicity (see [3]).

**Remark 6.3.10.** If  $(X, \tau)$  and  $(Y, \mu)$  for  $X, Y \subseteq M$  are definable topological spaces and  $f : (X, \tau) \rightarrow (Y, \mu)$  is a definable homeomorphism, then

- (i) if  $(X, \tau)$  contains an interval subspace with the discrete topology then  $(Y, \mu)$  contains an interval subspace with the discrete topology,
- (ii) if  $(X, \tau)$  contains an interval subspace with the right half-open or left half-open interval topology then  $(Y, \mu)$  contains an interval subspace with the right half-open or left half-open interval topology,
- (iii) if  $(X, \tau)$  contains an interval subspace with the euclidean topology then  $(Y, \mu)$  contains an interval subspace with the euclidean topology.

Note that (i) and (iii) hold for spaces of all dimensions if we substitute “interval subspace” with “subspace of dimension  $n$ ”.

Hence definable topological spaces in the line can be classified up to definable homeomorphism according to whether or not they contain interval subspaces with the euclidean,

discrete or half-open interval topologies. Moreover by Proposition 6.3.1 every infinite  $T_1$  space will fall into at least one of these categories.

#### 6.4 Hausdorff regular ( $T_3$ ) spaces. Decomposition in terms of the $\tau_{lex}$ and $\tau_{Alex}$ topologies.

In this section we study Hausdorff regular (i.e.  $T_3$ ) spaces in the line. The main result is Theorem 6.4.3, which states that any such space can be partitioned into a finite set and two definable open subspaces, one of which definably embeds into a space with the lexicographic order topology, and the other into a space which we label the Alexandrov  $n$ -line. In the next section we will use this result and its proof, as well as Theorem 6.3.9, to address universality questions in our setting.

We start by introducing the relevant topologies.

**Definition 6.4.1.** *Given  $X \subseteq M^n$  we denote by  $<_{lex}$  the lexicographic order on  $X$  and by  $(X, \tau_{lex})$  the topological space induced by  $<_{lex}$  on  $X$ . Clearly this space is definable whenever  $X$  is.*

We focus on the space  $(M \times \{0, \dots, n\}, \tau_{lex})$ . This space satisfies that all the points in  $M \times \{i\}$ , for  $0 < i < n$ , are isolated. Moreover, for any  $x \in M$ , a basis of open neighborhoods of  $\langle x, 0 \rangle$  is given by sets  $\langle x, 0 \rangle \cup (y, x) \times \{0, \dots, n\}$  for  $y < x$ , and a basis of open neighborhoods of  $\langle x, n \rangle$  is given by sets  $\langle x, n \rangle \cup (x, y) \times \{0, \dots, n\}$  for  $y > x$ .

**Definition 6.4.2** (Definable Alexandrov  $n$ -line). *Let  $\tau_{Alex}$  be the topology on  $M^2$  where all points in  $M^2 \setminus M \times \{0\}$  are isolated and, for any  $x \in M$ , a basis of open neighborhoods of  $\langle x, 0 \rangle$  is given by sets*

$$\{\langle x, 0 \rangle\} \cup ((z, y) \setminus \{x\}) \times M \text{ for } z < x < y.$$

*Then for any  $n > 0$  we call the space  $(M \times \{0, \dots, n-1\}, \tau_{Alex})$  the definable Alexandrov  $n$ -line.*

Note that, in particular,  $(M \times \{0\}, \tau_{lex}) = (M \times \{0\}, \tau_{Alex}) = (M \times \{0\}, \tau_e)$ .



We may now state Theorem 6.4.3. Lemmas 6.4.6 and 6.4.8 are the bulk of the proof. They are also used in Section 6.6 to prove that all regular Hausdorff definable topological spaces in the line can be definably Hausdorff compactified (Theorem 6.6.6).

**Theorem 6.4.3.** *Let  $(X, \tau)$ ,  $X \subseteq M$ , be a regular and Hausdorff definable topological space. Then there exist disjoint definable open sets  $Y, Z \subseteq X$  with  $X \setminus (Y \cup Z)$  finite, and  $n_Y, n_Z > 0$ , such that the following holds.*

1. *There exists a definable embedding  $h_Y : (Y, \tau) \hookrightarrow (\{0, \dots, n_Y\}, \tau_{lex})$ .*
2. *There exists a definable embedding  $h_Z : (Z, \tau) \hookrightarrow (\{0, \dots, n_Z\}, \tau_{Alex})$ .*

In Lemma 6.4.6 we construct a finite family  $\mathcal{X}_{\text{open}}$  of pairwise disjoint definable open subsets of  $X$  such that  $X \setminus \bigcup \mathcal{X}_{\text{open}}$  is finite. In Lemma 6.4.8 we construct, for every  $A \in \mathcal{X}_{\text{open}}$ , a set  $A^*$  of the form  $I_A \times \{0, \dots, n_A\}$  for some interval  $I_A$  and  $n_A$ , and a definable embedding  $h_A : (A, \tau) \hookrightarrow (A^*, \tau_A)$ , where  $\tau_A$  is either  $\tau_{lex}$  or  $\tau_{Alex}$ . The construction will be such that  $I_A \cap I_{A'} = \emptyset$  for distinct  $A, A' \in \mathcal{X}_{\text{open}}$ . Then  $Z$  will be the union of all the sets  $A$  in  $\mathcal{X}_{\text{open}}$  such that  $(A^*, \tau_A) = (A^*, \tau_{Alex})$ , and  $h_Z$  the union of the respective embeddings  $h_A$ . The set  $Y$  and embedding  $h_Y$  are constructed similarly from the remaining sets in  $\mathcal{X}_{\text{open}}$ .

Throughout this section and until the end of the proof of Theorem 6.4.3 we fix a definable definable topological space  $(X, \tau)$  with  $X \subseteq M$ .

We introduce an equivalence relation on  $X$  induced by the topology  $\tau$  defined as follows. Given  $x, y \in X$  we write  $x \sim_\tau y$  when one of the following holds:

- (i)  $x = y$ .
- (ii) There exists some  $z \in X$  such that  $\{x, y\} \cap E_z \neq \emptyset$  and, for all  $z \in X$ ,  $x \in E_z \Leftrightarrow y \in E_z$ .

This relation is clearly reflexive and symmetric, and one easily checks that it is transitive. Moreover by Proposition 2.2.18 (b) it is definable. For any  $x \in X$  we denote by  $[x]$  the equivalence class  $\{y \in X : y \sim_\tau x\}$ . We prove some preliminary facts regarding this relation.

**Lemma 6.4.4.** *If  $(X, \tau)$  is  $T_1$  then every equivalence class of  $\sim_\tau$  is finite.*

*Proof.* Let  $x \in X$ . If  $x \in X \setminus \bigcup_{y \in X} E_y$  then  $[x] = \{x\}$ . If there is some  $y \in X$  such that  $x \in E_y$  then, from the definition of  $\sim_\tau$ , it follows that  $[x] \subseteq E_y$ . If  $(X, \tau)$  is  $T_1$  then, by Lemma 2.2.20, the set  $E_y$  is finite, so  $[x]$  is finite.  $\square$

**Lemma 6.4.5.** *Let  $(X, \tau)$  be regular and Hausdorff. Then there exists a cofinite set  $X' \subseteq X$  with the following properties.*

- (a) *For any  $x \in X$ , either  $[x] \subseteq X'$  or  $[x] \cap X' = \emptyset$  ( $X'$  is compatible with  $\sim_\tau$ ).*
- (b) *For any  $x \in X'$  and  $y \in X$ , if  $x \in E_y$  then  $y \in [x]$ .*
- (c) *For any  $x \in X'$ ,  $E_x \subseteq [x]$ . In particular if  $E_x$  is nonempty then  $E_x = [x]$ .*

*Proof.* Let  $H = \{(x, y) \in X^2 : y \in E_x \text{ and } x \not\sim_\tau y\}$ . Let  $H_1$  and  $H_2$  be the projections of  $H$  onto the first and second coordinate respectively. We start by showing that these sets are finite. If  $H_2$  is finite then, by Hausdorffness (see Proposition 6.2.6 (b)),  $H_1$  is finite. Towards a contradiction we suppose that  $H_2$  is infinite. Let  $g : H_2 \rightarrow H_1$  be a function given by  $y \mapsto \min\{x : y \in E_x \text{ and } y \not\sim_\tau x\}$ . By Hausdorffness this function is well defined and by Lemma 2.2.20 it cannot be constant on an interval, so by o-minimality there exists  $I \subseteq H_2$  an interval such that  $g|_I$  is strictly monotonic and  $e$ -continuous. But then, by Lemma 6.2.12, for any  $y \in I$  it holds that  $y \sim_\tau g(y)$ , contradiction.

Set  $X' := X \setminus (\bigcup_{x \in H_1 \cup H_2} [x])$ . By Lemma 6.4.4 and finiteness of  $H_1 \cup H_2$  this set is cofinite in  $X$ . By definition of  $H$  it follows that  $X'$  satisfies (a)-(c).  $\square$

The next lemma strengthens Lemma 6.4.5 and is the core construction in the proofs of Theorems 6.4.3 and 6.6.6.

**Lemma 6.4.6.** *Let  $(X, \tau)$  be regular and Hausdorff and  $X' \subseteq X$  be as in Lemma 6.4.5. There exists a finite partition  $\mathcal{X}$  of  $X$  into singletons  $\mathcal{X}_{sgl}$  and infinite definable open sets  $\mathcal{X}_{open}$ , the latter being subsets of  $X'$ , with the following properties.*

*For every  $A \in \mathcal{X}_{open}$  there exists  $n > 0$ , an interval  $I \subseteq A$ , and definable  $e$ -continuous strictly monotonic functions  $f_0, f_1, \dots, f_{n-1} : I \rightarrow A$  such that, for every  $x \in I$ ,  $[x] = \{f_i(x) : 0 \leq i < n\}$ . In particular  $f_0$  is the identity map. Set  $I_i := f_i(I)$  for  $0 \leq i < n$ , then*

the intervals  $I_i$  are pairwise disjoint and  $A = \cup_{0 \leq i < n} I_i$ . Additionally, for every  $x \in I$  and  $0 < i < n - 1$ , the point  $f_i(x)$  is  $\tau$ -isolated.

Moreover; for every  $x \in I$  let  $[x]^E = \{y \in [x] : E_y \neq \emptyset\}$ . Then exactly one of the following conditions holds.

$$\forall x \in I \quad [x] = \{x\} \text{ and } E_x = \emptyset \text{ (} A = I \text{ contains only } \tau\text{-isolated points),}$$

$$\forall x \in I \quad [x]^E = \{x\},$$

$$\forall x \in I \quad [x]^E = \{x, f_{n-1}(x)\}, \text{ given } n > 1.$$

In each of the latter two cases exactly one of the following conditions is satisfied.

$$\forall x \in I \quad x \in L_x \setminus R_x,$$

$$\forall x \in I \quad x \in R_x \setminus L_x,$$

$$\forall x \in I \quad x \in R_x \cap L_x.$$

*Proof.* We construct  $\mathcal{X}$  by describing the family  $\mathcal{X}_{\text{open}}$  of open subsets of  $X'$ , while making sure that  $\cup \mathcal{X}_{\text{open}}$  is cofinite in  $X$ . In particular we consider a finite number of definable sets that partition  $X'$  and, for each such set  $S$ , describe a partition of a cofinite subset of  $S$ .

Let  $A_{\text{isol}} = X' \setminus \cup_{y \in X} E_y$ . Note that  $[x] = \{x\}$  for every  $x \in A_{\text{isol}}$ . By Lemma 6.4.5 (c), for all  $x \in A_{\text{isol}}$ ,  $E_x = \emptyset$ , and so, by Lemma 6.2.11, these points are  $\tau$ -isolated. If  $A_{\text{isol}}$  is infinite let  $\mathcal{X}_{\text{isol}}$  be a finite family of disjoint intervals whose union is cofinite in  $A_{\text{isol}}$ . Otherwise let  $\mathcal{X}_{\text{isol}} = \emptyset$ .

Now by Lemma 6.4.4 and uniform finiteness there exists  $n' \geq 1$  such that, for every  $x \in X$ ,  $|[x]| \leq n'$ . For every  $1 \leq n \leq n'$  set  $X_n := \{x \in X' \setminus A_{\text{isol}} : |[x]| = n\}$ . These sets are definable and partition  $X' \setminus A_{\text{isol}}$ . We fix  $1 \leq n \leq n'$ . If  $X_n$  is finite let  $\mathcal{X}_n = \emptyset$ . Suppose that  $X_n$  is infinite. We describe a finite partition  $\mathcal{X}_n$  of a cofinite subset of  $X_n$  into definable  $\tau$ -open sets as desired.

For every  $x \in X$  let  $[x]^E = \{y \in [x] : E_y \neq \emptyset\}$ . Since  $X' \setminus A_{\text{isol}} \subseteq \cup_{y \in X} E_y$  then, by Lemma 6.4.5 (b), for every  $x \in X_n$  it holds that  $|[x]^E| \geq 1$ . Moreover by Lemma 6.4.5 (c) and Hausdorffness (see Lemma 6.2.6 (b)), for every  $x \in X_n$  we have  $|[x]^E| \leq 2$ . Let  $X_n^{(1)} = \{x \in X_n : |[x]^E| = 1\}$  and  $X_n^{(2)} = \{x \in X_n : |[x]^E| = 2\}$ . These sets partition  $X_n$ .

Set  $Dom(X_n^{(1)}) := \cup\{[x]^E : x \in X_n^{(1)}\}$ ,  $Dom(X_n^{(2)}) := \{\min[x]^E : x \in X_n^{(2)}\}$  and  $Dom(X_n) := Dom(X_n^{(1)}) \cup Dom(X_n^{(2)})$ . By Lemma 6.4.5 (c) we may further partition  $Dom(X_n)$  into three definable sets as follows.

$$\begin{aligned} Dom(X_n)' &= \{x \in Dom(X_n) : x \in L_x \setminus R_x\}, \\ Dom(X_n)'' &= \{x \in Dom(X_n) : x \in R_x \setminus L_x\}, \\ Dom(X_n)''' &= \{x \in Dom(X_n) : x \in R_x \cap L_x\}. \end{aligned}$$

Now let  $f_0$  denote the identity map on  $X$ . Then, for  $1 \leq i < n$ , let  $f_i : Dom(X_n) \rightarrow X$  be the function defined as follows. For every  $x \in Dom(X_n^{(1)})$  and  $1 \leq i < n$ ,  $f_i(x)$  is the  $i$ -th smallest element in  $[x] \setminus \{x\}$ . For every  $x \in Dom(X_n^{(2)})$  and  $1 \leq i < n - 1$ ,  $f_i(x)$  is the  $i$ -th smallest element in  $[x] \setminus [x]^E$ , and  $f_{n-1}(x) = \max[x]^E$ . By construction, for every  $y \in X_n$ , there exists a unique  $x \sim_\tau y$  and  $0 \leq i < n$  such that  $f_i(x) = y$ . In particular  $[x] = \{f_i(x) : 0 \leq i < n\}$ , all functions  $f_i$  are injective and the family of images  $\{f_i(Dom(X_n)) : 0 \leq i < n\}$  is pairwise disjoint. Moreover, by construction, for every  $x \in Dom(X_n)$  and  $0 < i < n - 1$ ,  $E_{f_i(x)} = \emptyset$ , so, by Lemma 6.2.11,  $f_i(x)$  is  $\tau$ -isolated.

By o-minimality there exists a finite partition  $\mathcal{D}(X_n)$  of  $Dom(X_n)$ , compatible with  $\{Dom(X_n^{(1)}), Dom(X_n^{(2)}), Dom(X_n)', Dom(X_n)'', Dom(X_n)'''\}$ , which contains only singletons and intervals and such that, for every interval  $I \in \mathcal{D}(X_n)$ ,  $f_i|_I$  is  $e$ -continuous and strictly monotonic for every  $0 \leq i < n$ . The family of sets in  $\mathcal{X}_{\text{open}}$  which are subsets of  $X_n$  is then given by

$$\mathcal{X}_n = \{\cup_{0 \leq i < n} f_i(I) : I \in \mathcal{D}(X_n), I \text{ an interval}\}.$$

Note that, by construction,  $\cup_{0 \leq i < n} f_i(I) = \cup_{x \in I} [x]$  for every interval  $I \in \mathcal{D}(X_n)$ . Moreover, since the functions  $f_i$  are  $e$ -continuous, the sets  $f_i(I)$  are all intervals. So, using Lemma 6.4.5 (c) and Lemma 6.2.11, it is easy to see that the sets in  $\mathcal{X}_n$  are  $\tau$ -open.

Finally, set  $\mathcal{X}_{\text{open}} := \mathcal{X}_{\text{isol}} \cup \mathcal{X}_1 \cup \dots \cup \mathcal{X}_{n'}$ , and let  $\mathcal{X}_{\text{sgl}}$  denote the finite set of points in  $X \setminus \cup \mathcal{X}_{\text{open}}$  taken as singletons. By construction this partition satisfies the properties stated in the lemma. To check this, for any  $A \in \mathcal{X}_{\text{open}}$ , if  $A \subseteq X_n$  for some  $n$ , let  $I$  and  $f_i$ , for  $0 \leq i < n$ , be as above. If  $A \subseteq A_{\text{isol}}$  then simply consider  $I = A$  and  $f_0$ .  $\square$

Continuing with the construction in Lemma 6.4.6, the next lemma describes how each of the sets  $A \in \mathcal{X}_{\text{open}}$  definably embeds into a space with either the lexicographic or Alexandrov  $n$ -line topology.

We first require a definition expanding the notion of  $e$ -convergence from the right and from the left, and observing how this convergence relates to convergence with respect to the topologies  $\tau_{lex}$  and  $\tau_{Alex}$ .

**Definition 6.4.7.** *Given a definable set  $\tilde{X} \subseteq M \times \{0, 1, \dots\}$  we say that a definable curve in  $\tilde{X}$   $e$ -converges to  $\langle x, i \rangle \in \tilde{X}$  from the right (respectively left) if it  $e$ -converges to  $\langle x, i \rangle$  and its projection to the first coordinate, namely  $\pi \circ \gamma$ ,  $e$ -converges to  $x$  from the right (respectively left).*

Consider a definable set  $\tilde{X} = I \times \{0, \dots, n\}$ , with  $I \subseteq M$  an interval, and an injective definable curve  $\gamma$  in  $\tilde{X}$ . For  $x \in I$ , note that  $\gamma$  converges in  $(\tilde{X}, \tau_{lex})$  to  $\langle x, 0 \rangle$  if and only if it  $e$ -converges to  $\langle x, i \rangle$  from the left, for some  $0 \leq i \leq n$ ; and similarly it converges in  $(\tilde{X}, \tau_{lex})$  to  $\langle x, n \rangle$  if and only if it  $e$ -converges to  $\langle x, i \rangle$  from the right, for some  $0 \leq i \leq n$ . Moreover  $\gamma$  converges to  $\langle x, 0 \rangle$  in  $(\tilde{X}, \tau_{Alex})$  if and only if it  $e$ -converges to  $\langle x, i \rangle$  for some  $0 \leq i \leq n$  (from the right or from the left).

**Lemma 6.4.8.** *Let  $(X, \tau)$  be regular and Hausdorff and consider the construction of  $\mathcal{X}$  in Lemma 6.4.6. For each  $A = \cup_{0 \leq i < n} f_i(I) \in \mathcal{X}_{\text{open}}$  there exists a definable set  $A^* \subseteq M^2$  and a definable injection  $h_A : A \rightarrow A^*$  such that the following holds.*

(1)  $A^* = I \times \{0, \dots, m\}$ , for some  $m \in \{n-1, n, 2\}$  (in particular we have that, for every pair of distinct  $A_0, A_1 \in \mathcal{X}_{\text{open}}$ ,  $A_0^* \cap A_1^* = \emptyset$ ). Moreover, for every  $0 \leq i < n$  and  $x \in f_i(I)$ ,  $(\pi \circ h_A)(x) = f_i^{-1}(x)$ .

(2) The map  $h_A : (A, \tau) \hookrightarrow (A^*, \tau_A)$  is an embedding, where either

(a)  $(A^*, \tau_A) = (A^*, \tau_{lex})$  or

(b)  $(A^*, \tau_A) = (A^*, \tau_{Alex})$ .

*Proof.* Following Lemma 6.4.6, we distinguish different cases of possible  $A = \cup_{0 \leq i < n} f_i(I)$  in  $\mathcal{X}_{\text{open}}$ , based on properties of  $I$ . In each case we define  $A^*$  and  $h_A$  (which for simplicity we denote by  $h$ ) so that (1) and (2) hold.

**Case 0:**  $I = A$  is a set of isolated points in  $(X, \tau)$ .

Let  $A^* = A \times \{0, 1, 2\}$  and  $h : A \rightarrow A^*$  be given by  $x \mapsto \langle x, 1 \rangle$ . Let  $\tau_A$  be the topology induced by the lexicographic order on  $A^*$ . With this topology all the points in  $A \times \{1\}$  are isolated, so  $h$  is an embedding.

For the remaining cases we will make use of Corollary 6.2.3 to prove (3). After establishing whether  $(A^*, \tau_A) = (A^*, \tau_{lex})$  or  $(A^*, \tau_A) = (A^*, \tau_{Alex})$ , we will fix a point  $x \in X$  and prove that an injective definable curve  $\gamma$  converges in  $(A, \tau)$  to  $x$  if and only if  $h \circ \gamma$  converges in  $(A^*, \tau_A)$  to  $h(x)$ .

**Case 1:**  $[x]^E = \{x\}$  and  $x \in L_x \setminus R_x$  for every  $x \in I$ .

Let

$$A^* = I \times \{0, \dots, n\}$$

and let  $h : A \rightarrow A^*$  be the definable injection given by

$$h(f_i(x)) = \langle x, i \rangle$$

for every  $x \in I$  and  $0 \leq i < n$ .

Note that, for each  $0 \leq i < n$ ,  $h|_{I_i}$  is given by  $x \mapsto \langle f_i^{-1}(x), i \rangle$ , and so  $h$  is an  $e$ -embedding. We show that  $h$  is an embedding  $(A, \tau) \hookrightarrow (A^*, \tau_{lex})$ . Fix  $x \in A$ . If  $x \in I_i$  with  $i > 0$  then both  $x$  and  $h(x)$  are isolated, and hence not the limit of any injective definable curve, so we may assume that  $x \in I$ .

We will make use of the following observation, which follows from applying Lemma 6.2.12. For every  $0 \leq i < n$ ,

$$\begin{aligned} f_i(x) \in L_x &\Leftrightarrow f_i \text{ is increasing;} \\ f_i(x) \in R_x &\Leftrightarrow f_i \text{ is decreasing.} \end{aligned} \tag{\star}$$

Let  $\gamma$  be a definable injective curve in  $X$  that  $\tau$ -converges to  $x$ . Onwards recall Remark 6.2.10 and Definition 6.4.7. By properties of  $A$ ,  $\gamma$  must  $e$ -converge to some  $f_i(x)$ , from either the right or the left. Since  $f_i$  is an  $e$ -homeomorphism,  $h \circ \gamma$   $e$ -converges to  $\langle x, i \rangle$ . If  $\gamma$   $e$ -converges from the left then it must be that  $f_i(x) \in L_x$ , so by  $(\star)$   $f_i$  is increasing and

consequently  $h \circ \gamma$  also  $e$ -converges to  $\langle x, i \rangle$  from the left, therefore converging in  $(A^*, \tau_{lex})$  to  $\langle x, 0 \rangle = h(x)$ . Similarly if  $\gamma$   $e$ -converges from the right then it must be that  $f_i(x) \in R_x$ , and so by  $(\star)$   $f_i$  is decreasing and thus  $h \circ \gamma$   $e$ -converges to  $(x, i)$  from the left, so again it converges in  $(A^*, \tau_{lex})$  to  $\langle x, 0 \rangle = h(x)$ .

Conversely let  $\gamma' \subseteq h(A)$  be an injective definable curve converging in  $(A^*, \tau_{lex})$  to  $\langle x, 0 \rangle$ , in which case it must  $e$ -converge from the left to some  $\langle x, i \rangle$ . We may assume that  $\gamma' \subseteq I \times \{i\}$  (see Remark 2.2.12). Note that  $h^{-1} \circ \gamma' = f_i \circ \pi \circ \gamma'$ . If  $f_i$  is increasing then, by  $(\star)$ ,  $f_i(x) \in L_x$ , and moreover  $h^{-1} \circ \gamma'$   $e$ -converges to  $f_i(x)$  from the left, so it  $\tau$ -converges to  $x$ . Similarly if  $f_i$  is decreasing then, by  $(\star)$ ,  $f_i(x) \in R_x$ , and moreover  $h^{-1} \circ \gamma'$   $e$ -converges to  $f_i(x)$  from the right, so again it  $\tau$ -converges to  $x$ .

**Case 2:**  $[x]^E = \{x\}$  and  $x \in R_x \setminus L_x$  for every  $x \in I$ .

Let  $A^* = I \times \{0, \dots, n\}$  and  $h : A \rightarrow A^*$  be the definable injection given by  $h(f_i(x)) = \langle x, n - i \rangle$  for  $x \in I$  and  $0 \leq i < n$ . Again this is clearly an  $e$ -embedding. We show that  $h : (A, \tau) \hookrightarrow (A^*, \tau_{lex})$  is an embedding.

We proceed as in Case 1. Fix  $x \in A$ . Since otherwise the points  $x$  and  $h(x)$  are isolated in their respective spaces we may assume that  $x \in I$ . We make use of the next two equivalence following from Lemma 6.2.12. For every  $0 \leq i < n$ ,

$$\begin{aligned} f_i(x) \in R_x &\Leftrightarrow f_i \text{ is increasing ,} \\ f_i(x) \in L_x &\Leftrightarrow f_i \text{ is decreasing.} \end{aligned} \tag{6.2}$$

Let  $\gamma$  be an injective definable curve  $\tau$ -converging to  $x$ . Then  $\gamma$   $e$ -converges to some  $f_i(x)$ . If  $\gamma$   $e$ -converges from the right then it must be that  $f_i(x) \in R_x$ , so, by (6.2),  $f_i$  is increasing, and so  $h \circ \gamma$   $e$ -converges to  $h(f_i(x)) = (x, n - i)$  from the right, so it converges in  $(A^*, \tau_{lex})$  to  $(x, n) = h(x)$ . If it  $e$ -converges from the left then  $f_i(x) \in L_x$  and, again by (6.2),  $f_i$  is decreasing, so  $h \circ \gamma$   $e$ -converges to  $\langle x, n - i \rangle$  from the right, converging in  $(A^*, \tau_{lex})$  again to  $\langle x, n \rangle$ .

Conversely let  $\gamma' \subseteq h(A)$  be an injective definable curve that converges in  $(A^*, \tau_{lex})$  to  $h(x)$ . Then it must  $e$ -converge to some  $\langle x, j \rangle$  from the right. We may assume that  $\gamma' \subseteq I \times \{j\}$ . Let  $i = n - j$ . If  $f_i$  is increasing then, by (6.2),  $f_i(x) \in R_x$  and moreover

$h^{-1} \circ \gamma = f_i \circ \pi \circ \gamma'$   $e$ -converges to  $f_i(x)$  from the right, so it follows that it  $\tau$ -converges to  $x$ . Similarly If  $f_i$  is decreasing then, by (6.2),  $f_i(x) \in L_x$  and  $h^{-1} \circ \gamma$   $e$ -converges to  $f_i(x)$  from the left and so again it  $\tau$ -converges to  $x$ .

**Case 3:**  $n > 1$ ,  $[x]^E = \{x, f_{n-1}(x)\}$  and  $x \in L_x \setminus R_x$  for every  $x \in I$ .

Let  $A^* = I \times \{0, \dots, n-1\}$  and  $h : A \rightarrow A^*$  be defined as in case 1, namely  $h(f_i(x)) = \langle x, i \rangle$  for every  $x \in I$  and  $0 \leq i < n$ . In this case  $h : A \rightarrow A^*$  is a bijection. We show that  $h$  is a homeomorphism  $(A, \tau) \rightarrow (A^*, \tau_{lex})$ .

We fix  $y \in A$ . The case  $y \in I_i$ , for  $0 < i < n-1$  is as usual trivial. If  $y \in I$  then the result follows from the corresponding argument in case 1. Suppose that  $y \in I_{n-1}$ .

Let  $y = f_{n-1}(x)$ . Recall that, by Lemma 6.4.6 and Lemma 6.4.5 (c),  $E_x = E_y = \cup_i f_i(x)$ . By Hausdorffness it follows that, for every  $0 \leq i \leq n-1$ , exactly one of the following two possibilities holds

$$\begin{aligned} f_i(x) &\in R_x \cap L_y, \\ f_i(x) &\in R_y \cap L_x. \end{aligned} \tag{6.3}$$

Let  $\gamma$  be a definable injective curve  $\tau$ -converging  $y$ . Then  $\gamma$   $e$ -converges to some  $f_i(x)$ . Since  $h$  is an  $e$ -homeomorphism,  $h \circ \gamma$   $e$ -converges to  $\langle x, i \rangle$ , from the right or from the left. However, if  $h \circ \gamma$   $e$ -converges from the left – converging thus in  $(A^*, \tau_{lex})$  to  $\langle x, 0 \rangle$  – then, by continuity of  $h^{-1}$  at  $h(x) = \langle x, 0 \rangle$ , it must be that  $h^{-1} \circ h \circ \gamma = \gamma$   $\tau$ -converges to  $x$ , a contradiction (by Hausdorffness of  $\tau$ ). So  $h \circ \gamma$  must  $e$ -converge from the right, meaning that it converges in  $(A^*, \tau_{lex})$  to  $\langle x, n-1 \rangle = h(y)$ . Conversely let  $\gamma'$  be an injective definable curve converging in  $(A^*, \tau_{lex})$  to  $h(y) = \langle x, n-1 \rangle$ . Then it  $e$ -converges to some  $\langle x, i \rangle$ , meaning that  $h^{-1} \circ \gamma'$   $e$ -converges to  $f_i(x)$ . It  $h^{-1} \circ \gamma'$  does not  $\tau$ -converge to  $y$  then, by (6.3), it  $\tau$ -converges to  $x$ , but then by continuity of  $h$  at  $x$  it follows that  $\gamma'$  converges in  $(A^*, \tau_{lex})$  to  $\langle x, 0 \rangle$ , a contradiction. So  $h^{-1} \circ \gamma'$   $\tau$ -converges to  $y$ .

**Case 4:**  $n > 1$ ,  $[x]^E = \{x, f_{n-1}(x)\}$  and  $x \in R_x \setminus L_x$  for every  $x \in I$ .

Again let  $A^* = I \times \{0, \dots, n-1\}$  and  $h : A \rightarrow A^*$  be given as in case 2, namely by  $h(f_i(x)) = \langle x, n-i \rangle$ . Note that  $h$  is again a bijection. Moreover  $h : (A, \tau) \rightarrow (A^*, \tau_{lex})$  is a homeomorphism.



The proof follows from the proofs of the other cases. The argument in case 2 shows that, for any  $x \in A \setminus I_{n-1}$ , both  $h$  and  $h^{-1}$  are continuous at  $x$  and  $h(x)$  respectively. Then for the case of points in  $I_{n-1}$  one may use an argument analogous to the one in case 3.

**Case 5:**  $x \in R_x \cap L_x$  for every  $x \in I$ .

Set  $A^* := \bigcup_{0 \leq i \leq n-1} I \times \{i\}$  and let  $h : A \rightarrow A^*$  be given by  $h(f_i(x)) = \langle x, i \rangle$ . This map is clearly bijective. We show that it is a homeomorphism  $(A, \tau) \rightarrow (A^*, \tau_{Alex})$ .

Applying Lemma 6.2.12 we note that, in this case,  $f_i(x) \in R_x \cap L_x$  for every  $x \in I$  and  $0 \leq i < n$ . In particular by Hausdorffness and Lemma 6.4.5 (c)  $[x]^E = \{x\}$  for every  $x \in I$ , meaning that for every  $0 < i \leq n-1$  the point  $f_i(x)$  is  $\tau$ -isolated. Thus it remains to check continuity at points  $x$  and  $h(x)$  for  $x \in I$ .

Since  $R_x = L_x = [x]$  for any  $x \in I$  any injective definable curve converges in  $(A, \tau)$  to  $x$  if and only if it  $e$ -converges to some  $f_i(x)$ . Similarly any injective definable curve converges in  $(A^*, \tau_{Alex})$  to  $\langle x, 0 \rangle$  if and only if it  $e$ -converges to some  $\langle x, i \rangle$ . Thus the result follows from the fact that  $h$  is an  $e$ -homeomorphism, which is clear from the definition.

These are all the cases to consider, which completes the proof of the lemma.  $\square$

We may now prove Theorem 6.4.3.

*Proof of Theorem 6.4.3.* Let  $\mathcal{X}$  be a partition of  $X$  as given by Lemma 6.4.6 and, for each  $A \in \mathcal{X}_{\text{open}}$ , let  $A^*$ ,  $h_A$  and  $\tau_A$  be as given by Lemma 6.4.8.

Set  $h := \bigcup \{h_A : A \in \mathcal{X}_{\text{open}}\}$ . By Lemma 6.4.8 (1)  $h$  is an injection  $\bigcup \mathcal{X}_{\text{open}} \rightarrow \bigcup_{A \in \mathcal{X}_{\text{open}}} A^*$ . Let  $Y = \bigcup \{A \in \mathcal{X}_{\text{open}} : (A^*, \tau_A) = (A^*, \tau_{lex}) \neq (A^*, \tau_{Alex})\}$  and  $Z = \bigcup \{A \in \mathcal{X}_{\text{open}} : (A^*, \tau_A) = (A^*, \tau_{Alex})\}$ . By construction these sets are disjoint, open and definable, and  $X \setminus (Y \cup Z)$  is finite. Set  $Y^* := \bigcup \{A^* : A \in \mathcal{X}, A \subseteq Y\}$  and  $Z^* := \bigcup \{A^* : A \in \mathcal{X}, A \subseteq Z\}$ . We claim that  $h|_Y : (Y, \tau) \rightarrow (Y^*, \tau_{lex})$  and  $h|_Z : (Z, \tau) \rightarrow (Z^*, \tau_{Alex})$  are embeddings. We show that this claim holds by noting that we may decompose the maps into embeddings between open subspaces of their domains and codomains.

Recall that, by Lemma 6.4.8 (1), for each  $A \in \mathcal{X}$ , the set  $A^*$  is of the form  $I \times \{0, \dots, m\}$  for some  $m$  and interval  $I \subseteq A$ . It follows that, for each  $A \in \mathcal{X}$ , if  $A \subseteq Z$  then  $A^*$  is open in  $(Z^*, \tau_{Alex})$ . Similarly, if  $A \in Y$ , then  $A^*$  is open in  $(Y^*, \tau_{lex})$ , and the subspace topology

on  $A^*$  is precisely the topology induced by the lexicographic order on  $A^*$ . The claim then follows from Lemma 6.4.8 (2).

Finally, if  $Y \neq \emptyset$  let  $n_Y = \max\{n : (M \times \{n\}) \cap Y^* \neq \emptyset\}$ . It is easy to see that the map given by  $\langle x, m \rangle \mapsto \langle x, n_Y \rangle$  if  $m = \max\{m' : \langle x, m' \rangle \in Y^*\}$  and the identity otherwise is a definable embedding  $(Y^*, \tau_{lex}) \hookrightarrow (M \times \{0, \dots, n_Y\}, \tau_{lex})$ .  $\square$

**Remark 6.4.9.** If  $\mathcal{M}$  expands an ordered field then, by Remark 2.2.16, Theorem 6.4.3 applies to all  $T_3$  one-dimensional spaces. Otherwise the theorem and its proof may be rewritten to apply to one-dimensional spaces as follows.

Let  $(X, \tau)$  be a  $T_3$  one-dimensional definable topological space. In the context of Remark 6.2.13 one may prove analogs of Lemmas 6.4.4 and 6.4.5, and ultimately reach a partition  $\mathcal{X} = \mathcal{X}_{\text{open}} \cup \mathcal{X}_{\text{sgl}}$  of  $X$  as described in Lemma 6.4.6. Then, analogously to the proof of Lemma 6.4.8, it is possible to show that, for any  $A \in \mathcal{X}_{\text{open}}$ , there exists some  $m$ , an interval  $I$ , and a definable embedding  $h_A : (A, \tau) \hookrightarrow (A^*, \tau_A)$ , where  $A^* = \cup_{0 \leq i < m} I \times \{i\}$  and  $\tau_A \in \{\tau_{lex}, \tau_{Alex}\}$ . For any  $n > 0$  let  $X_n = M \times \{0, \dots, n-1\}$ . We may conclude that  $X$  has a cofinite subset definably homeomorphic to the disjoint union of finitely many spaces of the form  $(X_n, \tau_{lex})$  or  $(X_n, \tau_{Alex})$ .

In [42] Ramakrishnan showed that, if  $\mathcal{M}$  has elimination of imaginaries and defines an order reversing injection (e.g. if  $\mathcal{M}$  expands an ordered group), then every definable linear order definably embeds into  $(M^n, <_{lex})$  for some  $n$ . In particular, under these assumptions, for any definable order topological space one may assume that, up to definable homeomorphism, the topology is induced by the lexicographic order. Theorem 6.4.3 adds to the understanding of  $T_3$  definable spaces in the line by describing how the  $\tau_{Alex}$  topology also plays a role describing them. We complete this picture with the next proposition.

**Proposition 6.4.10.** *For any interval  $I$  and any  $n > 0$ , the space  $(I \times \{0, \dots, n\}, \tau_{Alex})$  does not definably embed into a definable order topological space.*

*Proof.* It suffices to prove the proposition for  $n = 1$ . Towards a contradiction assume there exists one such embedding into a space  $(X, \tau)$  where  $\tau$  is given by a definable linear order  $\preceq$ . Let  $Y$  denote the image of the aforementioned embedding.

Note that the subspace  $(I \times \{0\}, \tau_{Alex}) = (I \times \{0\}, \tau_e)$  is definably connected, and that the  $\tau_{Alex}$ -closure of any infinite definable subset of  $I \times \{0, 1\}$  intersects  $I \times \{0\}$ . It follows that any clopen definable subset of  $(I \times \{0, 1\}, \tau_{Alex})$  is either finite or cofinite. We reach a contradiction by showing that  $(Y, \tau)$  contains an infinite coinfinite definable clopen subset.

Let  $Y_1 = \{x \in Y : (x, +\infty)_{\preceq} \cap Y \text{ is finite}\}$ . Since the intervals  $(x, +\infty)_{\preceq}$  are nested note that, by uniform finiteness, the set  $Y_1$  is finite. Similarly the set  $Y_2 = \{x \in Y : (-\infty, x)_{\preceq} \cap Y \text{ is finite}\}$  is also finite.

Since  $(Y, \tau)$  is homeomorphic to  $(I \times \{0, 1\}, \tau_{Alex})$  and all the points in  $I \times \{1\}$  are  $\tau_{Alex}$ -isolated it follows that  $(Y, \tau)$  has infinitely many isolated points. Let us fix  $x \in Y$  an isolated point in  $(Y, \tau)$  that does not belong in  $Y_1 \cup Y_2$ . There must exist  $y \prec z$  in  $X$  such that

$$(y, z)_{\preceq} \cap Y = \{x\}.$$

It follows that the set  $(x, +\infty)_{\preceq} \cap Y = [z, +\infty)_{\preceq} \cap Y$  is clopen in  $(Y, \tau)$ . Since  $x \notin Y_1 \cup Y_2$  then it is also infinite and coinfinite in  $Y$ , contradiction.  $\square$

Finally, the next corollary of Theorem 6.4.3 implies that, for any  $T_3$  definably separable definable topological space  $(X, \tau)$ , where  $X \subseteq M$ , there exists a cofinite subset  $Y \subseteq X$  such that  $\tau|_Y$  is induced by a definable linear order.

**Corollary 6.4.11.** *Let  $(X, \tau)$ ,  $X \subseteq M$ , be a regular Hausdorff definable topological space. If  $(X, \tau)$  is definably separable then there exists a cofinite subset  $Y \subseteq X$ , a definable set  $Y^* \subseteq M \times \{0, 1\}$  and a definable homeomorphism  $(Y, \tau) \rightarrow (Y^*, \tau_{lex})$ .*

*Proof.* Recall all the notation from Lemmas 6.4.6 and 6.4.8. Our aim is to show that, under the additional assumption that  $(X, \tau)$  is definably separable, the construction in these lemmas yields that, for every  $A \in \mathcal{X}_{\text{open}}$ ,  $A^* \subseteq M \times \{0, 1\}$  and  $(A^*, \tau_A) = (A^*, \tau_{lex})$ . Then let  $Y = \cup \mathcal{X}_{\text{open}}$  and  $Y^* = \cup \{A^* : A \in \mathcal{X}_{\text{open}}\}$ . Following the proof of Theorem 6.4.3,  $Y$  is cofinite and  $h = \cup \{h_A : A \in \mathcal{X}_{\text{open}}\}$  is an embedding  $(Y, \tau) \hookrightarrow (Y^*, \tau_{lex})$ .

If  $(X, \tau)$  is definably separable it can have only finitely many isolated points. Following Lemma 6.4.6 fix  $A = \cup_{0 \leq i < n} I_i \in \mathcal{X}_{\text{open}}$ . For every  $x \in I$  and  $0 < i < n - 1$  the point  $f_i(x)$  is

$\tau$ -isolated. It follows that we must have  $n \leq 2$ . Similarly, for every  $x \in I$ , it must be that  $[x]^E = \{x, f_{n-1}(x)\}$ , since otherwise  $E_y = \emptyset$  (i.e.  $y$  is  $\tau$ -isolated) for every  $y \in I_{n-1}$ .

If  $n = 1$ , this is considered in Cases 1, 2 and 5 in the proof of Lemma 6.4.8. In Cases 1 and 2,  $A^* = I \times \{0, 1\}$  and  $(A^*, \tau_A) = (A^*, \tau_{lex})$ . In Case 5, we have that  $A^* = I \times \{0\}$  and  $(A^*, \tau_A) = (A^*, \tau_{Alex})$ , and in this case  $(A^*, \tau_{Alex}) = (A^*, \tau_{lex})$ . If  $n = 2$ , this is considered in Cases 3 and 4 in the proof of Lemma 6.4.8. In both these cases we have that  $A^* = I \times \{0, 1\}$  and  $(A^*, \tau_A) = (A^*, \tau_{lex})$ .  $\square$

## 6.5 Some universality results

In this section we answer, using results from the previous two sections, universality questions in the definable setting.

**Definition 6.5.1.** *Let  $\mathcal{C}$  be a class of definable topological spaces and let  $(X, \tau)$  be a definable topological space. We say that  $(X, \tau)$  is definably universal for  $\mathcal{C}$  if every space in  $\mathcal{C}$  embeds definably into  $(X, \tau)$ .*

*We say that  $(X, \tau)$  is almost definably universal for  $\mathcal{C}$  if, for every  $(Y, \mu) \in \mathcal{C}$  there exists a definable subset  $Z \subseteq Y$  with  $\dim(Y \setminus Z) < \dim Y$  such that  $(Z, \mu)$  embeds definably into  $(X, \tau)$ .*

Note that if a space is almost definably universal for a class then in particular it is definably universal.

We now observe how o-minimality implies that, if  $\mathcal{M}$  expands an ordered field,  $M^n$  is almost definably universal for the class of euclidean spaces of dimension at most  $n$ . An analogous results can be proved for the class of bounded euclidean spaces of dimension at most  $n$  when  $\mathcal{M}$  expands an ordered group.

**Proposition 6.5.2.** *Suppose that  $\mathcal{M}$  expands an ordered field. Then  $(M^n, \tau_e)$  is almost definably universal for the class of euclidean spaces of dimension less than or equal to  $n$ .*

*Proof.* Let  $(X, \tau)$  be a euclidean space with  $\dim X = n$ . Applying Remark 2.2.16 let  $(Y, \mu)$  denote its push-forward into  $M^n$  by some definable function  $f$ . We prove that  $(Y, \mu)$  contains a definable subspace  $Z \subseteq Y$  with  $\dim(Y \setminus Z) < \dim Y$  where the subspace topology is

euclidean. The case where  $\dim X = m < n$  follows from the fact that  $M^m$  embeds definably into  $M^n$ .

Recall that, by o-minimal cell decomposition, any definable bijection is a finite union of definable homeomorphisms in the euclidean topology. Applying this to  $f$ , it follows that  $Y$  can be partitioned into finitely many cells  $\mathcal{D}$  where the subspace topology is euclidean. Let  $Z = \cup\{int_\mu D : D \in \mathcal{D}, \dim D = n\}$ . By the frontier dimension inequality,  $\dim(Y \setminus Z) < \dim Y$ . Note that, since any cell of dimension  $n$  is open, for any  $D \in \mathcal{D}$  with  $\dim D = n$  the set  $int_\mu D$  is  $e$ -open as well as  $\tau$ -open in  $D$ . Moreover the subspace topology on  $int_\mu D$  is euclidean. We conclude that the subspace topology in  $Z$  is euclidean.  $\square$

**Remark 6.5.3.** In the general case where  $\mathcal{M}$  does not necessarily expand an ordered field one may still adapt the proof of Proposition 6.5.2 to show that, if  $(X, \tau)$  is a euclidean space of dimension  $n$  with cell decomposition  $\mathcal{D}$ , then the union  $Z$  of the interiors in  $X$  of cells in  $\mathcal{D}$  of dimension  $n$  embeds definably into finitely many disjoint copies of  $M^n$  (one for each cell). It follows that the space  $M^{n+1}$  is almost definably universal for the class of euclidean spaces of dimension at most  $n$ .

The question of definable universality for euclidean spaces is less straightforward. Chris Miller and Erik Walsberg proved (unpublished) a definable version of the classical Menger-Nöbeling theorem that implies that, whenever  $\mathcal{M}$  expands an ordered field, any euclidean space of dimension  $n$  embeds definably into  $M^{2n+1}$ .

Theorem 6.4.3 may be framed in terms of existence of an almost definably universal space as follows.

**Corollary 6.5.4.** *The disjoint union of  $(M \times [0, 1], \tau_{lex})$  and  $(M \times [0, \infty), \tau_{Alex})$  is Hausdorff, regular and almost definably universal for the class of Hausdorff regular definable topological spaces  $(X, \tau)$ , where  $X \subseteq M$ .*

*Proof.* It is easy to observe that the spaces  $(M \times [0, 1], \tau_{lex})$  and  $(M \times [0, \infty), \tau_{Alex})$  are Hausdorff and regular, from where it follows that the disjoint union is too.

Let  $(X, \tau)$ , where  $X \subseteq M$ , be a regular Hausdorff definable topological space. By Theorem 6.4.3 there exist definable disjoint open sets  $Y, Z \subseteq X$  and  $n_Y > 0$  such that

$X \setminus (Y \cup Z)$  is finite and there are definable embeddings  $(Y, \tau) \hookrightarrow (M \times \{0, \dots, n_Y\}, \tau_{lex})$  and  $(Z, \tau) \hookrightarrow (M \times [0, \infty), \tau_{Alex})$ . Hence it suffices to show that, for any  $n > 0$ , there exists a definable embedding  $(M \times \{0, \dots, n\}, \tau_{lex}) \hookrightarrow (M \times [0, 1], \tau_{lex})$ . Fix parameters  $0 = a_0 < a_1 < \dots < a_n = 1$ . Then the map given by  $\langle x, i \rangle \mapsto \langle x, a_i \rangle$  does the job.  $\square$

**Remark 6.5.5.** Similarly to Corollary 6.5.4, Theorem 6.4.3 and Proposition 6.4.10 may be used to show that the disjoint union of  $(M \times [0, 1], \tau_{lex})$  and  $(M, \tau_e)$  (note that the topology in this union is given by the lexicographic order) is almost definably universal for the class of definable topological spaces  $(X, \tau)$  with  $X \subseteq M$  that embed definably into a definable order topological space.

Onwards let  $\mathcal{C}_{\dim 1}^{T_3}$  denote the class of one-dimensional regular Hausdorff definable topological spaces.

**Remark 6.5.6.** If  $\mathcal{M}$  expands an ordered field then, by Remark 2.2.16, the space described in Corollary 6.5.4 is almost definably universal for  $\mathcal{C}_{\dim 1}^{T_3}$ .

In general, recall that Remark 6.4.9 states that any space in  $\mathcal{C}_{\dim 1}^{T_3}$  can be partitioned into finitely many points and open subsets definably homeomorphic to some set  $M \times \{0, \dots, n-1\}$  with either the  $\tau_{lex}$  or  $\tau_{Alex}$  topology. Consequently, following the arguments in the proof of Corollary 6.5.4, one may show that a space given by infinitely many copies of  $(M \times [0, 1], \tau_{lex})$  and  $(M \times [0, \infty), \tau_{Alex})$  is almost definably universal for  $\mathcal{C}_{\dim 1}^{T_3}$ . Such a space exists as a three-dimensional space. That is, consider  $(X, \tau)$ , where  $X = ((-\infty, 0) \times M \times [0, 1]) \cup ([0, \infty) \times M \times [0, \infty))$ . Then let  $\tau$  be the topology such that, for every  $t \in M$ , the fiber of  $X$ , which is given by either  $\{t\} \times M \times [0, 1]$  or  $\{t\} \times M \times [0, \infty)$ , is open and its projection to the last two coordinates is a homeomorphism onto  $(M \times [0, 1], \tau_{lex})$  or  $(M \times [0, \infty), \tau_{Alex})$  respectively.

Note that Proposition 6.5.2 states that, when  $\mathcal{M}$  expands an ordered field, the class of euclidean spaces of dimension at most  $n$  contains an almost definably universal space for itself (namely  $M^n$ ). In light of these results it is natural to ask if, in the case that  $\mathcal{M}$  expands an ordered field, there exists a space  $(X, \tau) \in \mathcal{C}_{\dim 1}^{T_3}$  that is almost definably universal for  $\mathcal{C}_{\dim 1}^{T_3}$  (note that the space in Corollary 6.5.4 is not one-dimensional), and more generally which classes of spaces admit an almost definably universal space, and when does the space

belong in the class. This sort of universality question is common in the study of Banach spaces [7] [11]. We answer the question regarding the class  $\mathcal{C}_{\dim 1}^{T_3}$  negatively and derive, from Theorem 6.3.9 and Corollary 6.4.11, positive answers for two other classes of one-dimensional spaces.

**Proposition 6.5.7.** *There does not exist a one-dimensional definable topological space  $(X, \tau)$  that is almost definably universal for  $\mathcal{C}_{\dim 1}^{T_3}$ .*

In order to prove the proposition we require a definition and two lemmas.

**Definition 6.5.8.** *Let  $(X, \tau)$  be a definable topological space. We say that two definable curves  $\gamma : (a, b) \rightarrow X$  and  $\mu : (a', b') \rightarrow X$ , with fixed domain endpoints  $c \in \{a, b\}$  and  $c' \in \{a', b'\}$  respectively, are equivalent if, for any definable topology  $\mu$  on  $X$ ,  $\mu\text{-}\lim_{t \rightarrow c} \gamma(t) = \mu\text{-}\lim_{t \rightarrow c'} \gamma'(t)$  whenever one of the two limits exists.*

*For any  $x \in X$  let  $\mathbf{n}(x, X, \tau)$  ( $\mathbf{n}(x)$  for short) denote the cardinality of a maximum set of non-equivalent definable curves in  $X$   $\tau$ -converging to  $x$ .*

*Let  $\mathbf{n}(X, \tau)$  ( $\mathbf{n}(X)$  for short) be defined to be  $\sup\{\mathbf{n}(x) : x \in X\}$ .*

If  $\dim X \leq 1$  and  $(X, \tau)$  is  $T_1$  then one may easily check (see Remarks 6.2.10 and 6.2.13) that  $\mathbf{n}(x) = 1 + |R_x| + |L_x|$ . By Lemma 2.2.20, Proposition 6.2.8 (a) and (b), and uniform finiteness it follows that  $\mathbf{n}(X) < \omega$ .

**Lemma 6.5.9.** *Let  $f : (X, \tau) \rightarrow (Y, \mu)$ , be a continuous injective definable map between definable topological spaces. For any  $x \in X$  it holds that  $\mathbf{n}(x) \leq \mathbf{n}(f(x))$ . In particular  $\mathbf{n}(X) \leq \mathbf{n}(Y)$ .*

*Proof.* By Corollary 6.2.3 if  $\gamma$  and  $\gamma'$  are two definable curves  $\tau$ -converging to  $x$  then  $f \circ \gamma$  and  $f \circ \gamma'$   $\mu$ -converge to  $f(x)$ . It therefore suffices to show that if  $\gamma$  and  $\gamma'$  are non-equivalent then  $f \circ \gamma$  and  $f \circ \gamma'$  are non-equivalent.

If  $\gamma$  and  $\gamma'$  are non-equivalent with domain  $(a, b)$  and  $(a', b')$  respectively and converging as, say,  $t \rightarrow a$  and  $t \rightarrow a'$  respectively, then there exists  $a < d < b$  and  $a' < d' < b'$  such that  $\gamma[(a, d)] \cap \gamma'[(a', d')] = \emptyset$ . By injectivity  $(f \circ \gamma)[(a, d)] \cap (f \circ \gamma')[(a', d')] = \emptyset$ , and so  $f \circ \gamma$  and  $f \circ \gamma'$  are clearly non-equivalent.  $\square$

A number of observations can be derived from Lemma 6.5.9 regarding the existence of definably universal spaces. For a start, note that, given  $n$ , if  $L(i)$  denotes the line  $\{0\} \times \cdots \times \{0\} \times M \times \{0\} \times \cdots \times \{0\}$ , where  $M$  is in the  $i$ -th coordinate position, then the euclidean space  $\cup_{1 \leq i \leq n} L(i) \subseteq M^n$  satisfies that  $\mathbf{n}(0, \dots, 0) = 2n$ . Hence, by Lemma 6.5.9, there does not exist a one-dimensional  $T_1$  definable topological space that is definably universal for all such spaces. This shows that there does not exist one such space definably universal for the class of one-dimensional euclidean spaces (and so neither for  $\mathcal{C}_{\dim 1}^{T_3}$ ).

**Lemma 6.5.10.** *Let  $\mathcal{C}$  be a class of  $T_1$  definable topological spaces and let  $(X, \tau)$  be a definable topological space. If  $(X, \tau)$  is definably universal (respectively almost definably universal) for  $\mathcal{C}$ , then there exists a  $T_1$  definable topology  $\mu$  finer than or equal to  $\tau$  such that the space  $(X, \mu)$  is definably universal (respectively almost definably universal) for  $\mathcal{C}$ .*

*Proof.* Let  $(X, \tau)$  be a definable topological space. For every  $x \in X$  let  $C_x = (cl_\tau\{x\}) \setminus \{x\}$ . Note that this set is uniformly definable over  $x \in X$ . If  $\mathcal{U}$  denotes a definable basis for  $\tau$  let  $\mathcal{U}' = \{U \setminus C_x : U \in \mathcal{U}, x \in X\}$ . We claim that the definable family  $\mathcal{U}'$  is a basis for a topology  $\mu$ , which will clearly be  $T_1$  and finer than or equal to  $\tau$ .

Now given the claim suppose that there exists  $Y \subseteq X$  such that  $(Y, \tau)$  is  $T_1$ . Then, for every  $x \in Y$ ,  $C_x \cap Y = \emptyset$ . So, for any  $x \in X$ , if  $C_x \cap Y \neq \emptyset$  then  $x \notin Y$ , meaning  $Y \setminus C_x = Y \setminus cl_\tau\{x\}$ , which is an open set in  $(Y, \tau)$ . In particular any  $U' \in \mathcal{U}'$  satisfies that  $U' \cap Y$  is open in  $(Y, \tau)$ . It follows that  $(Y, \tau) = (Y, \mu)$ . We derive that, if  $(X, \tau)$  is definably universal (or almost definably universal) for a class of  $T_1$  spaces, then  $(X, \mu)$  is too, proving the lemma. It remains to prove the claim.

Fix  $U_1, U_2 \in \mathcal{U}$  and  $x_1, x_2 \in X$ . For simplicity, for  $i \in \{1, 2\}$ , set  $C_i = C_{x_i}$ . Let  $x \in (U_1 \setminus C_1) \cap (U_2 \setminus C_2)$ . We must find  $U \in \mathcal{U}'$  such that  $x \in U \subseteq (U_1 \setminus C_1) \cap (U_2 \setminus C_2) = U_1 \cap U_2 \setminus (C_1 \cup C_2)$ . Since  $\mathcal{U}$  is a basis there exists  $U_0 \in \mathcal{U}$  such that  $x \in U_0 \subseteq U_1 \cap U_2$ , and it suffices to find  $U \in \mathcal{U}'$  such that  $x \in U \subseteq U_0 \setminus (C_1 \cup C_2)$ .

If  $x \notin cl_\tau(C_1 \cup C_2)$ , then the existence of one such set is immediate. Otherwise  $x \in cl_\tau(C_1 \cup C_2) = (cl_\tau C_1) \cup (cl_\tau C_2)$ . Without loss of generality suppose  $x \in cl_\tau C_1$ , in which case  $x \in cl_\tau\{x_1\}$ . However, since  $x \notin C_1$ , it must be that  $x = x_1$ . Now, if  $x \notin cl_\tau C_2$ , then we may assume that  $U_0$  satisfies that  $U_0 \cap C_2 = \emptyset$ , in which case it is enough to consider  $U = U_0 \setminus C_1$ .



If  $x \in cl_\tau C_2$  then, by the same argument as before, we have that  $x = x_2$ . Hence  $x = x_1 = x_2$  and once again we can take  $U = U_0 \setminus C_1 = U_0 \setminus C_2$ .  $\square$

We may now prove Proposition 6.5.7.

*Proof of Proposition 6.5.7.* Let  $Y = M \times \{0, \dots, n-1\}$  and consider the space  $(Y, \tau_{lex})$ , which belongs in  $\mathcal{C}_{\dim 1}^{T_3}$ . If, for every  $0 \leq i < n$ , we identify the subspace  $M \times \{i\}$  with  $M$  through the projection to the first coordinate (see Remark 6.2.13) then, for any  $x \in M$ , it holds that  $L_{\langle x, 0 \rangle} = \{\langle x, 0 \rangle, \dots, \langle x, n-1 \rangle\}$  and  $R_{\langle x, 0 \rangle} = \emptyset$ . So  $\mathbf{n}(x, 0) = n$ , and in fact  $\mathbf{n}(Y) = n$ . Moreover note that, for any cofinite subset  $Y' \subseteq Y$ , it still holds that  $\mathbf{n}(Y) = n$ , since we may always find an interval  $I \subseteq M$  such that  $I \times \{0, \dots, n-1\} \subseteq Y'$ .

Suppose that  $(X, \tau)$  is a one-dimensional definable topological space that is almost definably universal for  $\mathcal{C}_{\dim 1}^{T_3}$ . By Lemma 6.5.10 we may assume that  $(X, \tau)$  is  $T_1$ . By Lemma 6.5.9 and the above observation we have that  $\mathbf{n}(X) \geq n$  for every  $n$ , contradiction.  $\square$

The same proof would have still be worked considering the space  $(M \times \{0, \dots, n-1\}, \tau_{Alex})$  in place of  $(M \times \{0, \dots, n-1\}, \tau_{lex})$ . Ultimately one may show that there exists no one-dimensional definable topological space that is almost definably universal for either of the following two classes: all one-dimensional spaces with the  $\tau_{lex}$  topology and all one-dimensional spaces with the  $\tau_{Alex}$  topology.

Nevertheless, Theorem 6.4.3 – more specifically Corollary 6.4.11 – does yield the existence of a certain class of one-dimensional spaces that contains a space which is almost definably universal for the class, as shown by the following corollary.

Let  $\mathcal{C}_{\dim 1}^{T_3, sep}$  denote the class of Hausdorff regular definably separable one-dimensional spaces.

**Corollary 6.5.11.** *The disjoint union of  $(M, \tau_e)$  and  $(M \times \{0, 1\}, \tau_{lex})$  is Hausdorff, regular, definably separable, and almost definably universal for the class of Hausdorff, regular, definably separable spaces  $(X, \tau)$ , where  $X \subseteq M$ .*

*It follows that, whenever  $\mathcal{M}$  expands an ordered field, the class  $\mathcal{C}_{\dim 1}^{T_3, sep}$  contains an almost definably universal space.*

*Proof.* The second paragraph of the corollary follows from the first by direct application of Remark 2.2.16. We prove the first paragraph.

Since  $(M, \tau_e)$  and  $(M \times \{0, 1\}, \tau_{lex})$  are regular, Hausdorff and definably separable their disjoint union is too.

By Corollary 6.4.11 it suffices to show that, for any definable set  $X \subseteq M \times \{0, 1\}$ , there exists a cofinite subspace  $Y$  of  $(X, \tau_{lex})$  that embeds definably in the disjoint union of  $(M, \tau_e)$  and  $(M \times \{0, 1\}, \tau_{lex})$ .

We partition  $X \subseteq M \times \{0, 1\}$  as follows. Let  $X_1 = \{\langle x, i \rangle \in X : \langle x, 1 - i \rangle \notin X\}$  and  $X_2 = X \setminus X_1$ . By  $\mathfrak{o}$ -minimality there exists a partition  $\mathcal{X}$  of a cofinite subset of  $X$  with the following properties. For every  $A \in \mathcal{X}$ , there exists an interval  $I$  such that either  $A = I \times \{i\}$ , for some  $i \in \{0, 1\}$ , and  $A \subseteq X_1$ , or  $A = I \times \{0, 1\}$  and  $A \subseteq X_2$ . Let  $\mathcal{X}_1 = \{A \in \mathcal{X} : A \subseteq X_1\}$  and  $\mathcal{X}_2 = \mathcal{X} \setminus \mathcal{X}_1$ .

Note that every  $A \in \mathcal{X}$  is open in  $(X, \tau_{lex})$ , and that the subspace topology on  $A$  corresponds precisely to the lexicographic order topology on  $A$ . If  $A \subseteq X_1$  then the map  $\langle x, i \rangle \mapsto x$  is an open embedding  $(A, \tau_{lex}) \hookrightarrow (M, \tau_e)$ , and otherwise the identity is an open embedding  $(A, \tau_{lex}) \hookrightarrow (M \times \{0, 1\}, \tau_{lex})$ . Hence the projection to the first coordinate is an open embedding  $(\cup \mathcal{X}_1, \tau_{lex}) \hookrightarrow (M, \tau_e)$  and the identity is an open embedding  $(\cup \mathcal{X}_2, \tau_{lex}) \hookrightarrow (M \times \{0, 1\}, \tau_{lex})$ , which completes the proof.  $\square$

Following Remarks 6.5.3 and 6.5.6, if  $\mathcal{M}$  does not expand a ordered field then one may adapt the proof of Corollary 6.5.11 to show that every space in  $\mathcal{C}_{\dim 1}^{T_3, \text{sep}}$  has a cofinite subspace that embeds definably into finitely many disjoint copies of  $(M, \tau_e)$  and  $(M \times \{0, 1\}, \tau_{lex})$ , and so one may construct a two-dimensional space that is almost definably universal for  $\mathcal{C}_{\dim 1}^{T_3, \text{sep}}$ .

Note that any definable subspace of  $(M, \tau_e)$  and  $(M \times \{0, 1\}, \tau_{lex})$  is definably separable. By Corollary 6.5.11 and the above paragraph it follows that any definable subspace of a space in  $\mathcal{C}_{\dim 1}^{T_3, \text{sep}}$  is also definably separable. In other words, definable separability is a hereditary property for  $T_3$  one-dimensional spaces.

Finally we consider the class of one-dimensional Hausdorff definable topological spaces with the frontier dimension inequality (f.d.i.), which we denote  $\mathcal{C}_{\dim 1}^{T_2, \text{fdi}}$ . By Proposition 6.2.14

we know that these spaces are regular. We show that, whenever  $\mathcal{M}$  expands an ordered field, the class contains an almost definably universal space. It is a corollary of Theorem 6.3.9.

**Corollary 6.5.12.** *The disjoint union of spaces  $(M, \tau_e)$ ,  $(M, \tau_r)$ ,  $(M, \tau_l)$  and  $(M, \tau_s)$  is Hausdorff, satisfies the frontier dimension inequality, and is almost definably universal for the class of Hausdorff definable topological spaces with the frontier dimension inequality  $(X, \tau)$  where  $X \subseteq M$ .*

*It follows that, whenever  $\mathcal{M}$  expands an ordered field, the class  $\mathcal{C}_{\dim 1}^{T_2, fdi}$  contains an almost definably universal space.*

*Proof.* The second paragraph of the corollary follows from the first by direct application of Remark 2.2.16. We prove the first paragraph.

Since the spaces  $(M, \tau_e)$ ,  $(M, \tau_r)$ ,  $(M, \tau_l)$  and  $(M, \tau_s)$  are Hausdorff and have the f.d.i., their disjoint union has these properties too.

We fix  $(X, \tau)$ ,  $X \subseteq M$ , a Hausdorff definable space with the f.d.i. By Theorem 6.3.9, there exists a finite partition  $\mathcal{X}$  of  $X$  into points and intervals such that, for each  $I \in \mathcal{X}$ , the subspace topology  $\tau|_I$  is one of  $\tau_e$ ,  $\tau_r$ ,  $\tau_l$  or  $\tau_s$ . Let  $\mathcal{X}'$  be the subfamily of intervals in  $\mathcal{X}$  and let  $X' = \bigcup\{int_\tau I : I \in \mathcal{X}'\}$ . By the frontier dimension inequality the set  $X'$  is cofinite in  $X$ . Partition  $X'$  into four definable sets as follows. Let  $X_e = \{x \in X' : x \in I \in \mathcal{X}, (I, \tau) = (I, \tau_e)\}$ . Then the identity is an open embedding  $(X_e, \tau) \hookrightarrow (M, \tau_e)$ . By repeating this argument with the topologies  $\tau_r$ ,  $\tau_l$  and  $\tau_s$  we may conclude that  $X'$  can be partitioned into four definable open subspaces on which the identity is an embedding into one of  $(M, \tau_e)$ ,  $(M, \tau_r)$ ,  $(M, \tau_l)$  or  $(M, \tau_s)$ . The corollary follows.  $\square$

Following Remarks 6.5.3 and 6.5.6, if  $\mathcal{M}$  does not expand a ordered field then one may adapt the proof of Corollary 6.5.12 to show that every space in  $\mathcal{C}_{\dim 1}^{T_2, fdi}$  has a cofinite subspace that embeds definably into finitely many disjoint copies of  $(M, \tau_e)$ ,  $(M, \tau_r)$ ,  $(M, \tau_l)$  and  $(M, \tau_s)$ , and so one may construct a two-dimensional space that is almost definably universal for  $\mathcal{C}_{\dim 1}^{T_2, fdi}$ .

## 6.6 Definable Hausdorff compactifications

In this section we address the question of which definable topological spaces can be Hausdorff compactified in a definable sense, concluding (Theorem 6.6.6) that these include all regular Hausdorff spaces in the line.

Recall that two definable curves  $\gamma : (a, b) \rightarrow M^n$  and  $\gamma' : (a', b') \rightarrow M^n$  with fixed extreme points  $c \in \{a, b\}$  and  $c' \in \{a', b'\}$  respectively are equivalent (Definition 6.5.8) if, given any definable topological space  $(X, \tau)$  and  $x \in X$ ,  $\gamma$   $\tau$ -converges to  $x$  if and only if  $\gamma'$   $\tau$ -converges to  $x$ . If  $c = a$  and  $c' = a'$  then this is equivalent to having that, for every  $a < d < b$ , there is  $a' < d' < b'$  such that  $\gamma'[(a', d')] \subseteq \gamma[(a, d)]$ .

**Definition 6.6.1.** *A definable topological space  $(X, \tau)$ ,  $\dim X \leq 1$ , is definably near-compact if, up to equivalence, there are only finitely many non-convergent definable curves in  $(X, \tau)$ .*

*Clearly definable compactness implies definable near-compactness. We say that a definable topological space  $(X^*, \tau^*)$ ,  $X^* \subseteq M$ , is a definable near-compactification of  $(X, \tau)$  if the former is definably near-compact and the latter definably embeds into the former.*

**Lemma 6.6.2.** *A definable topological space  $(X, \tau)$ , where  $X \subseteq M$ , is definably near-compact if and only if the set  $(\cup_{x \in X} R_x) \cap (\cup_{x \in X} L_x)$  is cofinite in  $cl_e X$ .*

*Proof.* This is a direct consequence of Remark 6.2.10 and the fact that, by o-minimality, every injective definable curve in  $M$   $e$ -converges to some point in  $M_{\pm\infty}$  from the right or from the left. □

**Remark 6.6.3.** Note that, for any  $n$  and interval  $I$ , the spaces  $(I \times \{0, \dots, n\}, \tau_{lex})$  and  $(I \times \{0, \dots, n\}, \tau_{Alex})$  are definably near-compact. It follows that the embedding  $h : (Y \cup Z, \tau) \hookrightarrow (Y^* \cup Z^*, \tau_{lex}|_{Y^*} \cup \tau_{Alex}|_{Z^*})$  described in the proof of Theorem 6.4.3 is a definable near-compactification of a cofinite open subspace of  $X$ .

We extract the following observation from the proof of Lemma 6.4.8.

**Proposition 6.6.4.** *Let  $(X, \tau)$  be regular and Hausdorff. Let  $\mathcal{X}$  be as in Lemma 6.4.6. For each  $A \in \mathcal{X}_{open}$ , let  $A^*$  and  $h_A : A \rightarrow A^*$  be as in Lemma 6.4.8.*

If  $(X, \tau)$  is definably near-compact then, for any  $A \in \mathcal{X}_{open}$ , the map  $h_A$  is a bijection. In particular, by Lemma 6.4.8(2), it is a definable homeomorphism  $(A, \tau) \rightarrow (A^*, \mu)$ , where  $\mu$  is one of  $\tau_{lex}$  or  $\tau_{Alex}$ .

*Proof.* Recall the proof by cases of Lemma 6.4.8. Observe that, by Lemma 6.4.5 (b), if  $(X, \tau)$  is definably near-compact then, for all but finitely many  $x \in I$ , it holds that  $x \in R_y \cap L_z$ , for some  $y, z \in [x]^E \subseteq \{x, f_{n-1}(x)\}$ . It follows that, if  $(X, \tau)$  is definably near-compact, cases 0, 1 and 2 in the proof of Lemma 6.4.8 are not possible. In the remaining cases the function  $h_A$  defined is a bijection.  $\square$

The idea behind the definition of definable near-compactness is that it is the property that characterizes the one-dimensional  $T_3$  spaces that can be one-point definably Hausdorff compactified. We prove one direction of this characterization in the following proposition.

**Proposition 6.6.5.** *Let  $(X, \tau)$ ,  $\dim X \leq 1$ , be a regular Hausdorff definably near-compact space. Then there exists a Hausdorff definably compact space  $(X^c, \tau^c)$  and a definable embedding  $h : (X, \tau) \hookrightarrow (X^c, \tau^c)$ , where  $X^c \setminus h(X)$  is a singleton.*

If  $\mathcal{M}$  expands the field of reals then  $(X^c, \tau^c)$  is simply the one-point compactification of  $(X, \tau)$ .

*Proof of Proposition 6.6.5.* We prove the lemma in the case where  $X \subseteq M$ . Given the assumptions in Remark 6.2.13 the proof adapts to a proof of the general case.

Let  $c = \langle 0, 1 \rangle$  and  $h$  be the map on  $X$  given by  $x \mapsto \langle x, 0 \rangle$ . Let  $X^c = X \times \{0\} \cup \{c\}$  and  $\tau_h$  be the push-forward topology of  $\tau$  by  $h$  (see Definition 2.2.15). We will define  $\tau^c$  as an expansion of  $\tau_h$  to a topology on  $X^c$ .

Set  $R_c := \{x \in M_{\pm\infty} \setminus \cup_{x \in X} R_x : \exists y > x (x, y) \subseteq X\}$  and  $L_c := \{x \in M_{\pm\infty} \setminus \cup_{x \in X} L_x : \exists y < x (y, x) \subseteq X\}$ . Set  $E_c := R_c \cup L_c$ . Since  $(X, \tau)$  is definably near-compact,  $E_c$  is finite. Let  $R_c = \{y_1, \dots, y_n\}$  and  $L_c = \{z_1, \dots, z_m\}$ , and let  $\mathcal{U}(c)$  be the family of sets

$$\bigcup_{1 \leq i \leq n} (y_i, y'_i) \cup \bigcup_{1 \leq j \leq m} (z'_j, z_j)$$

definable uniformly over parameters  $(y'_1, \dots, y'_n, z'_1, \dots, z'_m) \in M^{n+m}$ , where  $y_i < y'_i$  and  $z'_j < z_j$ . By definition of  $R_c$  and  $L_c$  we may moreover impose that the sets in  $\mathcal{U}(c)$  are all subsets of  $X$ .

Let  $\tau^c$  be the definable topology with basis  $\{(int_\tau U \times \{0\}) \cup \{c\} : U \in \mathcal{U}(c)\} \cup \tau_h$ . It is routine to check that this is a well-defined topology and that  $h : (X, \tau) \hookrightarrow (X^c, \tau^c)$  is an embedding. By definition of  $\mathcal{U}(c)$  and Lemma 6.2.11 it is immediate that  $(X, \tau^c)$  is Hausdorff. It remains to prove that it is definably compact.

Let  $\gamma'$  be a definable curve in  $(X^c, \tau^c)$ . We may assume that  $\gamma'$  is injective and hence lies in  $X \times \{0\}$ . Let  $\gamma = h^{-1} \circ \gamma'$ . Let  $x_0 \in M_{\pm\infty}$  denote the limit of  $\gamma$  in the euclidean topology. Since the remaining case is analogous we consider only the case where  $\gamma$   $e$ -converges to  $x_0$  from the right. Then clearly there must exist  $y > x_0$  such that  $(x_0, y) \subseteq X$ . By Remark 6.2.10, if  $x_0 \notin R_c$  then  $x_0 \in \cup_{x \in X} R_x$  and so  $\gamma$   $\tau$ -converges to some  $x \in X$ , and it follows that  $\gamma'$   $\tau^c$ -converges to  $h(x)$ . Suppose that  $x_0 \in R_c$ . We will show that  $\gamma'$   $\tau^c$ -converges to  $c$ . To prove this it suffices to show that, for every  $U \in \mathcal{U}(c)$ , there is  $x_U > x_0$  such that  $(x_0, x_U) \subseteq int_\tau U$ .

Towards a contradiction, suppose otherwise. Then, by o-minimality, there exists  $U_1 \in \mathcal{U}(c)$  and  $x_1 > x_0$  such that  $(x_0, x_1) \cap int_\tau U_1 = \emptyset$ . By definition of  $\mathcal{U}(c)$  we may moreover take  $x_1$  close enough to  $x_0$  to satisfy that  $(x_0, x_1) \subseteq U_1$ . For every  $x_0 < x < x_1$ ,  $x \in \partial_\tau(X \setminus U_1)$ , and so from Proposition 6.2.5 (b) it follows that  $E_x \setminus U_1 \neq \emptyset$ . Let  $f : (x_0, x_1) \rightarrow M_{\pm\infty}$  be the definable map given by  $x \mapsto \min E_x \setminus U_1$ . By Hausdorffness (Lemma 6.2.6 (b)) and o-minimality this function is  $e$ -continuous and strictly monotone on some subinterval  $(x_0, x_2) \subseteq (x_0, x_1)$ . Let  $y_0 = e - \lim_{x \rightarrow x_0} f(x)$ . If  $f$  is increasing on  $(x_0, x_2)$  then, by construction of  $\mathcal{U}(c)$  and the fact that  $f$  maps into  $M_{\pm\infty} \setminus U_1$ , it cannot be that  $y_0 \in R_c$ . However, there clearly exists  $y' \in M$  such that  $(y_0, y') \subseteq X$ . So there exists  $y \in X$  such that  $y_0 \in R_y$  and, by Lemma 6.2.12 and regularity, it follows that  $x_0 \in R_y$ , a contradiction since  $x_0 \in R_c$ . The case where  $f$  is decreasing is analogous.  $\square$

We now present the main result of this section.

**Theorem 6.6.6.** *Let  $(X, \tau)$ ,  $X \subseteq M$ , be a Hausdorff definable topological space. Then  $(X, \tau)$  is regular if and only if there exists a definably compact Hausdorff definable topological space  $(X^c, \tau^c)$ , with  $\dim X^c \leq 1$ , and a definable embedding  $(X, \tau) \hookrightarrow (X^c, \tau^c)$ .*

*Proof.* Let  $(X, \tau)$  be a Hausdorff definable topological space in the line. Since the finite case is trivial we assume that  $\dim X = 1$ . The “if” implication of the theorem follows directly from Lemma 5.4.7, Proposition 6.2.4 and the observation that regularity is a hereditary property. We prove the “only if” implication. Hence assume that  $(X, \tau)$  is regular. We will make use of Lemmas 6.4.6 and 6.4.8 to construct for  $(X, \tau)$  a one-dimensional definable Hausdorff compactification. It is perhaps interesting to note, although we will not use it in the proof, that one may show that the closure of any one-dimensional subset of a Hausdorff space is one-dimensional. Consequently, by passing if necessary to the closure of the image of the embedding, one may always assume that the definable Hausdorff compactification of a one-dimensional space is one-dimensional.

Recall the embedding  $(Y \cup Z, \tau) \hookrightarrow (Y^* \cup Z^*, \tau_{lex}|_{Y^*} \cup \tau_{Alex}|_{Z^*})$  described in the proof of Theorem 6.4.3. As noted in Remark 6.6.3, this embedding is a definable near-compactification of a cofinite open subspace of  $X$ . The idea of the current proof is to extend this embedding to an embedding of  $(X, \tau)$  into a regular Hausdorff definably near-compact space  $(X^*, \tau^*)$ , with  $X^* \subseteq M \times \{0, 1, \dots\}$ . Then Proposition 6.6.5 completes the proof.

Let  $\mathcal{X} = \mathcal{X}_{\text{open}} \cup \mathcal{X}_{\text{sgl}}$  be a finite partition of  $X$  as in Lemma 6.4.6. For each  $A \in \mathcal{X}_{\text{open}}$  let  $A^*$ ,  $\tau_A$  and  $h_A$  be as in Lemma 6.4.8. In particular recall that each set  $A^*$  is of the form  $I \times \{0, \dots, n\}$  for some  $n$  and interval  $I \subseteq A$ , and that  $\tau_A$  is either the  $\tau_{lex}$  or  $\tau_{Alex}$  topology on  $A^*$ . Moreover  $h_A : (A, \tau) \hookrightarrow (A^*, \tau_A)$  is a definable embedding.

Let  $X^* = \bigcup \{A^* : A \in \mathcal{X}_{\text{open}}\} \cup \{\langle x, 0 \rangle : \{x\} \in \mathcal{X}_{\text{sgl}}\}$  and  $h = h' \cup \bigcup \{h_A : A \in \mathcal{X}_{\text{open}}\}$ , where  $h'$  is the map with domain  $\bigcup \mathcal{X}_{\text{sgl}}$  given by  $x \mapsto \langle x, 0 \rangle$ . Note that  $h$  is injective.

We construct a regular Hausdorff topology  $\tau^*$  on  $X^*$  such that, for every  $A \in \mathcal{X}_{\text{open}}$ ,  $A^*$  is  $\tau^*$ -open and  $(A^*, \tau^*) = (A^*, \tau_A)$ . Since every space  $(A^*, \tau_A)$  is definably near-compact, it follows that  $(X^*, \tau^*)$  is definably near-compact. We then prove that  $h : (X, \tau) \hookrightarrow (X^*, \tau^*)$  is an embedding.

Let  $F = \cup \mathcal{X}_{\text{sgl}}$  and  $s = |\mathcal{X}_{\text{open}}|$ . We define  $\tau^*$  as follows. For every  $x \in F$  and  $A \in \mathcal{X}_{\text{open}}$  we construct a downward directed definable family  $\mathcal{B}_A(x)$  of  $\tau_A$ -open subsets of  $A^*$ . Then set, for each  $x \in F$ ,

$$\mathcal{B}(x) := \{ \{ \langle x, 0 \rangle \} \cup V_1 \cup \dots \cup V_s : (V_1, \dots, V_s) \in \prod_{A \in \mathcal{X}_{\text{open}}} \mathcal{B}_A(x) \}.$$

It is routine to check that then the family  $\cup \{ \mathcal{B}(x) : x \in F \} \cup \cup \{ \tau_A : A \in \mathcal{X}_{\text{open}} \}$  is a basis for a topology  $\tau^*$  on  $X^*$ , which will clearly satisfy that  $\tau^*|_{A^*} = \tau_A$  for every  $A \in \mathcal{X}_{\text{open}}$ . Since  $F$  is finite and the topologies  $\tau_A$  are definable,  $\tau^*$  is also definable.

We fix  $x \in F$  and  $A \in \mathcal{X}_{\text{open}}$  and describe  $\mathcal{B}_A(x)$ . Recall the notation  $A = \cup_{0 \leq i < n} I_i$  from Lemma 6.4.6, and let  $I_0 = I = (a, b)$ . For any  $y \in I$  let

$$V_a(x, y) = \begin{cases} ((a, y) \times M) \cap A^* & \text{if } a \in R_x, \\ \emptyset & \text{otherwise,} \end{cases}$$

$$V_b(x, y) = \begin{cases} ((y, b) \times M) \cap A^* & \text{if } b \in L_x, \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that these sets are always open in  $(A^*, \tau_A)$ . Then  $\mathcal{B}_A(x)$  is defined to be the family of sets

$$V_A(x, y, z) = V_a(x, y) \cup V_b(x, z),$$

definable uniformly in  $y$  and  $z$  with  $a < y < z < b$ . Clearly  $\mathcal{B}_A(x)$  is a definable downward directed family of open subsets of  $A^*$ . Since  $\cap_{a < y < z < b} V_A(x, y, z) = \emptyset$  it is immediate from the definition that the induced topology  $\tau^*$  is  $T_1$ . It remains to check that  $(X^*, \tau^*)$  is Hausdorff and regular and that  $h : (X, \tau) \hookrightarrow (X^*, \tau^*)$  is an embedding.

Consider the sets  $V_A(x, y, z)$  for some  $a < y < z < b$ . By Hausdorffness of  $\tau$ , for any two distinct  $x, x' \in F$ , if  $a \in R_x$  then  $a \notin R_{x'}$  and if  $b \in L_x$  then  $b \notin L_{x'}$ , so  $V_A(x, y, z) \cap V_A(x', y, z) = \emptyset$ , from where it follows that  $cl_{\tau^*} V_A(x, y, z) \subseteq \{ \langle x, 0 \rangle \} \cup A^*$ , and consequently  $cl_{\tau^*} V_A(x, y, z) = \{ \langle x, 0 \rangle \} \cup cl_{\tau_A} V_A(x, y, z)$ . Moreover note that, since  $\tau_A$  is one of  $\tau_{lex}$  or  $\tau_{Alex}$ , it holds that  $cl_{\tau_A} V_a(x, y) \subseteq ((a, y] \times M) \cap A^*$  and  $cl_{\tau_A} V_b(x, z) \subseteq ([z, b) \times M) \cap A^*$ . So, for any



$a < y' < y$  and  $z < z' < b$ , we have that  $cl_{\tau_A} V_A(x, y', z') \subseteq V_A(x, y, z)$ . It follows that, for every  $U \in \mathcal{B}(x)$ , there exists  $U' \in \mathcal{B}(x)$  such that  $cl_{\tau^*} U' \subseteq U$ . Since each  $A^*$  is  $\tau^*$ -open with  $(A^*, \tau^*) = (A^*, \tau_A)$ , it is easy to check that the same property holds among  $\tau^*$ -neighborhoods of points in  $A^*$ . It follows that the topology  $\tau^*$  is regular. In particular, since it is  $T_1$ , it is Hausdorff.

It remains to show that  $h : (X, \tau) \hookrightarrow (X^*, \tau^*)$  is an embedding. We fix  $x \in X$  and show continuity of  $h$  and  $h^{-1}$  at  $x$  and  $h(x)$  respectively. Since, for every  $A \in \mathcal{X}_{\text{open}}$ ,  $h_A : (A, \tau) \hookrightarrow (A^*, \tau_A)$  is an embedding between open subsets of  $X$  and  $X^*$  respectively, where  $\tau^*|_{A^*} = \tau_A$ , this holds whenever  $x \in A$  for some  $A \in \mathcal{X}_{\text{open}}$ , so we may assume that  $x \in F$ . We make use of Corollary 6.2.3.

Fix  $\gamma$  an injective definable curve in  $X$ , and set  $\gamma' := h \circ \gamma$ . Let  $A = \cup_{0 \leq i < n} I_i \in \mathcal{X}_{\text{open}}$  and  $0 \leq j < n$  be such that we may assume that  $\gamma$  is contained in the interval  $I_j$ . Recall from Lemma 6.4.6 that, for every  $0 \leq i < n$ ,  $I_i = f_i(I)$  for some definable  $e$ -continuous strictly monotonic function  $f_i : I_0 \rightarrow M$ . We will prove the case where  $f_j$  is increasing. The decreasing case is analogous. Let  $I_0 = I = (a, b)$  and  $I_j = (a_j, b_j)$ . We require the following simple fact that follows from the definition of the definable families  $\mathcal{B}_A(x)$ .

**Fact 6.6.7.** *The curve  $\gamma'$   $\tau^*$ -converges to  $h(x) = \langle x, 0 \rangle$  if and only if either  $a \in R_x$  and  $\pi \circ \gamma'$   $e$ -converges to  $a$ , or  $b \in L_x$  and  $\pi \circ \gamma'$   $e$ -converges to  $b$ .*

Suppose that  $\gamma$   $\tau$ -converges to  $x$ . Recall that, by Lemmas 6.4.5 (b) and 6.4.6, if  $A \cap E_x \neq \emptyset$ , then  $x \in A$ , a contradiction. So, by o-minimality and Proposition 6.2.5 (a),  $\gamma$  must  $e$ -converge to either  $a_j$  or  $b_j$ . Suppose that  $\gamma$   $e$ -converges to  $a_j$ , in which case  $a_j \in R_x$ . Since  $f_j$  is increasing, we have that  $f_j^{-1} \circ \gamma$   $e$ -converges to  $a$ . By regularity of  $\tau$  and Lemmas 6.2.12 and 6.4.6, it follows that  $a \in R_x$ . Now note that, by Lemma 6.4.8 (1),  $\pi \circ \gamma' = \pi \circ h \circ \gamma = f_j^{-1} \circ \gamma$ . Hence  $\pi \circ \gamma'$   $e$ -converges to  $a$ . So, by Fact 6.6.7, we conclude that  $\gamma'$   $\tau^*$ -converges to  $h(x) = \langle x, 0 \rangle$ . Analogously, if  $\gamma$   $e$ -converges to  $b_j$ , then  $b_j \in L_x$  and, again by regularity and Lemmas 6.2.12 and 6.4.6,  $b \in L_x$ . Moreover, by Lemma 6.4.8 (1),  $\pi \circ \gamma'$   $e$ -converges to  $b$ , so by Fact 6.6.7  $\gamma'$   $\tau^*$ -converges to  $h(x) = \langle x, 0 \rangle$ .

Now suppose that  $\gamma'$   $\tau^*$ -converges to  $h(x)$  and recall Fact 6.6.7. If  $\pi \circ \gamma$   $e$ -converges to  $a$ , then  $a \in R_x$  and, since  $f_j$  is increasing, by Lemmas 6.2.12 and 6.4.6, we have that  $a_j \in R_x$ .

Moreover, by Lemma 6.4.8 (1) and since  $f_j$  is increasing,  $\gamma = h^{-1} \circ \gamma' = f_j \circ \pi \circ \gamma'$   $e$ -converges to  $a_j$ . We conclude that  $h^{-1} \circ \gamma'$   $\tau$ -converges to  $x$ . Similarly, if  $\pi \circ \gamma$   $e$ -converges to  $b$ , then  $b \in L_x$  and so, by Lemmas 6.2.12 and 6.4.6,  $b_j \in L_x$ . Moreover  $\gamma$   $e$ -converges to  $b_j$ , so  $\gamma$   $\tau$ -converges to  $x$ . This completes the proof of the theorem.  $\square$

## 6.7 Affine one-dimensional topologies

Unless stated otherwise throughout this section we assume that our underlying structure expands an ordered field. Consequently we move to using  $\mathcal{R}$  and  $R$  in place of  $\mathcal{M}$  and  $M$  respectively. Recall that a definable topological space is affine if it is definably homeomorphic to a set with the euclidean topology. In this section we wish to classify affine one-dimensional definable topologies. Our main result is the following.

**Theorem 6.7.1.** *Suppose that  $\mathcal{R}$  expands an ordered field. Let  $(X, \tau)$ ,  $\dim X \leq 1$ , be a Hausdorff definable topological space. Exactly one of the following holds:*

- (1)  $(X, \tau)$  contains a subspace definably homeomorphic to an interval with either the discrete or the right half-open interval topology.
- (2)  $(X, \tau)$  is affine.

Note that, since the map  $x \mapsto -x$  is a homeomorphism  $(R, \tau_r) \rightarrow (R, \tau_l)$ , Theorem 6.7.1 still holds if we consider  $\tau_l$  instead of the  $\tau_r$  topology in (1).

Recall that, for any interval  $I \subseteq R$ , the space  $(I, \mu)$ , where  $\mu \in \{\tau_r, \tau_s\}$ , is totally definably disconnected (i.e. singletons are the only definably connected nonempty subspaces). On the other hand, by o-minimality, every euclidean space has finitely many definably connected components. Hence Theorem 6.7.1 implies that a Hausdorff one-dimensional definable topological space  $(X, \tau)$  is affine if and only if every definable subspace has finitely many definably connected components (or is not totally definably disconnected).

The main theorem (Theorem 7.1) in [49] states that if a definable metric space contains no infinite definable discrete subspace (equivalently, by Remark 3.1.6, if it is definably separable) then it is affine. Hence, taking into account this result, statement (2) in Theorem 6.7.1 can be changed to “ $(X, \tau)$  is definably separable and definably metrizable”.

A definition, a remark and a lemma precede the proof of Theorem 6.7.1.

**Definition 6.7.2.** *We say that a definable topological space  $(X, \tau)$  is cell-wise euclidean if it can be partitioned into finitely many cells  $\mathcal{X}$  such that, for each  $C \in \mathcal{X}$ ,  $(C, \tau) = (C, \tau_e)$ .*

By o-minimal cell decomposition we can clearly relax the condition that sets in  $\mathcal{X}$  in Definition 6.7.2 are cells to just any definable sets. Since every definable bijection is a finite union of disjoint definable  $e$ -homeomorphisms, the property of being cell-wise euclidean is maintained by definable homeomorphism, and is equivalent to being cell-wise affine. In particular any affine space is cell-wise euclidean. Theorem 6.7.1 implies that the converse holds for one-dimensional Hausdorff spaces. This statement cannot be generalized to spaces of all dimensions, as illustrated by Example A.16, which describes a two-dimensional definable topological space that is Hausdorff and cell-wise euclidean but not definably metrizable.

**Remark 6.7.3.** By o-minimal cell decomposition and Theorem 6.3.9, if a Hausdorff definable topological space  $(X, \tau)$ , with  $\dim X \leq 1$ , does not have a subspace that is definably homeomorphic to an interval with the  $\tau_r$  or  $\tau_s$  topologies, then  $(X, \tau)$  is cell-wise euclidean.

To see this, it suffices to note that, for any cell  $C \subseteq X$  and definable bijection  $\pi_C : C \rightarrow I \subseteq R$ , if  $(C, \tau)$  does not have a definable copy of an interval with the  $\tau_r$  or  $\tau_s$  topologies then, by Theorem 6.3.9, the push-forward of  $(C, \tau)$  by  $\pi_C$  must be cell-wise euclidean, and so  $(C, \tau)$  is cell-wise euclidean.

By the above remark, in order to prove Theorem 6.7.1 it suffices to show that Hausdorff one-dimensional spaces that are cell-wise euclidean are moreover affine.

The following lemma is essentially Lemma 5.7 in [49], which we extend to one-dimensional spaces using Propositions 6.2.4 and 5.4.5(4). Recall that a euclidean space is definably compact if and only if it is closed and bounded. This is easy to prove for one-dimensional spaces, and was proved for spaces of all dimensions in [37].

**Lemma 6.7.4.** *Let  $(X, \tau)$  be a definably compact Hausdorff definable topological space. Let  $(Y, \tau_e)$  be a definably compact euclidean space that admits a definable continuous surjection  $f : (Y, \tau_e) \rightarrow (X, \tau)$ . Then there exists a definable set  $Z$  and a definable homeomorphism  $(Z, \tau_e) \rightarrow (X, \tau)$ .*

*Proof.* Let  $E$  be the kernel of  $f$ , namely  $E = \{\langle x, y \rangle \in Y^2 : f(x) = f(y)\}$ . By continuity of  $f$ ,  $E$  is closed in  $Y^2$ , and so definably compact. By Chapter 10 (Corollary 2.16) in [17] there exists a definable set  $Z$  and a definable quotient map  $g : (Y, \tau_e) \rightarrow (Z, \tau_e)$  of  $E$ , i.e.  $g$  has kernel  $E$ , is surjective, continuous and, for every  $C \subseteq Z$ , if  $g^{-1}(C)$  is closed in  $(Y, \tau_e)$  then  $C$  is closed in  $(Z, \tau_e)$ . Moreover, by Proposition 5.4.5 (3), the space  $(Z, \tau_e)$  is definably compact.

The definable map  $h : (Z, \tau_e) \rightarrow (X, \tau)$  given by  $h(g(x)) = f(x)$  is clearly continuous and bijective. By Proposition 5.4.5(4) it is a homeomorphism.  $\square$

We may now prove Theorem 6.7.1.

*Proof of Theorem 6.7.1.* Let  $(X, \tau)$  be a Hausdorff definable topological space. The case where  $\dim(X) = 0$  is trivial and so we assume that  $\dim(X) = 1$ . By Remark 2.2.16 we assume that  $X \subseteq R$  and is bounded. Note that, by Remark 6.1.1,  $(X, \tau)$  cannot be both cell-wise euclidean and have a definable copy of an interval with the  $\tau_r$  or  $\tau_s$  topologies, so (1) and (2) in the statement of the theorem are mutually exclusive. Applying Remark 6.7.3, we assume that  $(X, \tau)$  is cell-wise euclidean and derive that it is affine.

Since  $(X, \tau)$  is cell-wise euclidean it is also definably near-compact so, by Lemma 6.6.5, by passing to  $(X^c, \tau^c)$  if necessary we may assume that  $(X, \tau)$  is definably compact.

Let  $\mathcal{X}$  be a partition of  $X$  into points and intervals such that, for each  $C \in \mathcal{X}$ , the subspace  $(C, \tau)$  is euclidean. We define, for each  $C \in \mathcal{X}$ , a continuous function  $f_C : (cl_e C, \tau_e) \rightarrow (cl_\tau C, \tau)$  extending the identity on  $C$ .

Once we have defined these functions we complete the proof as follows. Let  $num : \mathcal{X} \rightarrow \omega$  be a numbering of the elements in  $\mathcal{X}$  and  $Y = \cup_{C \in \mathcal{X}} cl_e C \times \{num(C)\}$  be the disjoint union of the euclidean closures of the sets in  $\mathcal{X}$ . Clearly  $(Y, \tau_e)$  is definably compact. Let  $f : (Y, \tau_e) \rightarrow (X, \tau)$  be the function given by  $f(x, i) = f_C(x)$ , where  $num(C) = i$ . This function is clearly surjective and continuous. By Lemma 6.7.4 the proof is complete. It remains to define, for each  $C \in \mathcal{X}$ , the function  $f_C$ .

If  $C \in \mathcal{X}$  is a singleton let  $f_C$  be simply the identity. Now let us fix an interval  $C = I = (a_I, b_I) \in \mathcal{X}$ . By Hausdorffness (Proposition 6.2.8 (c)) and definable compactness (Remark 6.2.10) there exists a unique point  $x_I \in X$  such that  $a_I \in R_{x_I}$  and similarly a

unique point  $y_I \in X$  such that  $b_I \in L_{y_I}$ . Note that, since  $(I, \tau) = (I, \tau_e)$ , the points  $x_I$  and  $y_I$  do not belong in  $I$ . Let  $f_I$  be defined as

$$f_I|_I = id, f(a_I) = x_I \text{ and } f(b_I) = y_I.$$

It is routine to check that  $f_I$  is continuous as a map  $([a_I, b_I], \tau_e) \rightarrow (X, \tau)$ . □

In Example A.14 we describe a Hausdorff definable topological space of dimension two that has no definable copy of an interval with the  $\tau_s$  or  $\tau_r$  topology but fails to be cell-wise euclidean. In Example A.16 we describe a regular Hausdorff definable topological space of dimension two that is cell-wise euclidean but not affine. Hence, although equivalent for one and zero-dimensional spaces, the following three implications are strict in general.

$$\begin{array}{ccccc} \text{Affine} & \Rightarrow & \text{Hausdorff and} & \Rightarrow & \text{Hausdorff and does not contain} \\ & & \text{cell-wise euclidean} & & \text{a definable copy of an interval} \\ & & & & \text{with either the } \tau_r \text{ or } \tau_s \text{ topology} \end{array}$$

This complicates the task of generalising Theorem 6.7.1 to spaces of all dimensions. The next corollary however offers a possibility.

**Corollary 6.7.5.** *Suppose that  $\mathcal{R}$  expands an ordered field. Let  $(X, \tau)$ ,  $\dim X \leq 1$ , be a definably compact Hausdorff definable topological space. The following are equivalent.*

- (1)  $(X, \tau)$  satisfies the frontier dimension inequality.
- (2)  $(X, \tau)$  is definably metrizable.
- (3)  $(X, \tau)$  is affine.

*Proof.* We fix  $(X, \tau)$ ,  $\dim X \leq 1$ , a definably compact Hausdorff definable topological space.

(3)  $\Rightarrow$  (2) is trivial. If  $(X, \tau)$  is definably metrizable then, by Lemma 7.15 in [49], it satisfies the f.d.i., i.e. (2)  $\Rightarrow$  (1). We complete the proof by showing (1)  $\Rightarrow$  (3), that is, if  $(X, \tau)$  satisfies the f.d.i., then it is affine.

Suppose that there exists an interval with the  $\tau_r$  or  $\tau_s$  topology that embeds definably into  $(X, \tau)$ . We prove that  $(X, \tau)$  does not have the f.d.i. By Theorem 6.7.1 this completes the proof.

By Remark 2.2.16 we may assume that  $X \subseteq R$ . By Remark 6.3.10 there exists an interval  $I \subseteq X$  such that  $\tau|_I \in \{\tau_r, \tau_l, \tau_s\}$ . Considering the push-forward of  $(X, \tau)$  by  $x \mapsto -x$  if necessary, we may assume that  $\tau|_I \in \{\tau_r, \tau_s\}$ . By definable compactness and Hausdorffness, for every  $y \in I$  there exists a unique  $x \in X$  such that  $y \in L_x$  (see Proposition 6.2.8 (c) and Remark 6.2.10). Since  $\tau|_I \in \{\tau_r, \tau_s\}$ , this  $x$  must belong in  $\partial_\tau I$ . By Lemma 2.2.20 it follows that  $\partial_\tau I$  is infinite, and so  $(X, \tau)$  does not have the f.d.i.  $\square$

## 6.8 Definable metrizable

In this section we explore the notion of definable metrizable (Definition 2.2.6) among one-dimensional spaces. We maintain the assumption that our underlying o-minimal structure  $\mathcal{R}$  expands an ordered field. Recall that, in our setting, “metric” refers to an  $\mathcal{R}$ -metric (Definition 2.2.4), including those instances when it appears implicitly in notions such as metrizable and metric space. Recall the definition of weight,  $w_\tau(X)$ , of a topological space  $(X, \tau)$ , i.e. the minimum cardinality of a basis for  $\tau$ . We denote the density of  $(X, \tau)$  by  $den_\tau(X)$ . Our main result, Theorem 6.8.2, shows that, whenever  $\mathcal{R}$  satisfies that  $den_e(R) < |R|$  (i.e. whenever  $\mathcal{R}$  expands the field of reals), every metrizable one-dimensional space is in fact definably metrizable.

Note that from Theorem 6.7.1 and Remark 6.3.4 (and since  $\mathcal{R}$  expands an ordered field) it follows that every one-dimensional Hausdorff definable topological space  $(X, \tau)$  satisfies that  $w_\tau(X) \in \{w_e(R), |R|\}$ . Moreover, from Theorem 6.7.1 we may derive the next corollary.

**Corollary 6.8.1.** *Let  $(X, \tau)$ ,  $\dim(X) \leq 1$ , be a definable topological space that is metrizable and separable. Suppose that any of the following two conditions hold.*

- $\mathcal{R}$  expands the field of reals.
- $(X, \tau)$  is compact.

*Then  $(X, \tau)$  is affine. In particular it is definably metrizable.*

*Proof.* The case where  $X$  is finite is trivial so we assume that  $\dim(X) = 1$ . By Remark 6.3.6, which states that if there exists a compact infinite  $T_1$  definable topological space then  $\mathcal{R}$  expands the field of reals, we may assume that  $\mathcal{R}$  expands the field of reals.

By Remark 2.2.16 we assume that  $X \subseteq R$ . By Theorem 6.7.1, it is enough to show that  $(X, \tau)$  does not have a definable copy of an interval with the discrete or right half-open interval topology. This follows from the fact that  $(X, \tau)$  is a separable metric space, hence second countable, so  $w(X) < 2^\omega$ , while the topological weight of an interval with the Sorgenfrey Line or discrete topology is  $2^\omega$  (Remark 6.3.4).  $\square$

We now state the main theorem of this section, which improves the metrization part of Corollary 6.8.1.

**Theorem 6.8.2.** *Suppose that  $\mathcal{R}$  expands an ordered field and satisfies  $den_e(R) < |R|$ . Let  $(X, \tau)$ ,  $\dim X \leq 1$ , be a definable topological space. Then  $(X, \tau)$  is metrizable if and only if it is definably metrizable.*

In order to prove Theorem 6.8.2 we require two simple lemmas, whose aim is to generalize basic results in metric topology and topology of the real line to our setting. In what follows recall that, since  $\mathcal{R}$  expands an ordered field, any two intervals are definably  $e$ -homeomorphic and in particular, for any interval  $I \subseteq R$ , we have  $|I| = |R|$  and  $den_e(I) = den_e(R)$ .

**Lemma 6.8.3.** *Let  $(X, d)$  be a metric space. Let  $\tau := \tau_d$ , and let  $A, B \subseteq X$  satisfy that  $A \subseteq cl_\tau(B)$  (i.e.  $B$  is  $\tau$ -dense in  $A$ ). Let  $D$  be an  $e$ -dense subset of  $(0, +\infty)$ . Consider the family of  $d$ -balls  $\mathcal{B} = \{B_d(y, \delta) : y \in B, \delta \in D\}$ . Then, for every  $x \in A$ , there exists a subfamily  $\mathcal{B}_x$  of  $\mathcal{B}$  that is a basis of open  $\tau$ -neighborhoods of  $x$ .*

*In particular  $w_\tau(X) \leq den_\tau(X)den_e(R)$ .*

*Proof.* Clearly every set in  $\mathcal{B}$  is  $\tau$ -open. Without loss of generality, fix  $x \in A$  and  $\varepsilon > 0$ . We must show that there exists  $y \in B$  and  $\delta \in D$  such that  $x \in B_d(y, \delta) \subseteq B_d(x, \varepsilon)$ .

Let  $\delta \in D$  be such that  $0 < \delta < \varepsilon/2$ . Since  $A \subseteq cl_\tau(B)$ , there exists  $y \in B$  such that  $d(x, y) < \delta$ . Consider the ball  $B_d(y, \delta)$ . Clearly  $x \in B_d(y, \delta)$  and, if  $z \in B_d(y, \delta)$ , then, by the triangle inequality,  $d(x, z) \leq d(x, y) + d(y, z) \leq \delta + \delta < \varepsilon$ . Hence  $x \in B_d(y, \delta) \subseteq B_d(x, \varepsilon)$ , which completes the proof of the first part of the lemma.

For the second part, suppose that  $B$  is a dense subset of  $(X, \tau)$  of cardinality  $den_\tau(X)$  and  $D$  is an  $e$ -dense subset of  $(0, \infty)$  of cardinality  $den_e(R)$ . Then, by the above, the family

$\{B_d(y, \delta) : y \in B, \delta \in D\}$  is a basis for  $\tau$  which has cardinality bounded by  $den_\tau(X)den_e(R)$ . □

**Remark 6.8.4.** Let  $X \subseteq R$  be an infinite definable set. By Remark 6.3.4 and because  $\mathcal{R}$  expands an ordered field the space  $(X, \tau_*)$ , where  $\tau_* \in \{\tau_l, \tau_r\}$ , has weight  $|R|$ . Moreover, clearly the density of  $(X, \tau_*)$  is equal to  $den_e(R)$ .

From Lemma 6.8.3 it follows that, if  $(X, \tau_*)$  is metrizable, then  $|R| = w_{\tau_*}(X) \leq den_{\tau_*}(X)den_e(R) = den_e(R)^2 = den_e(R) \leq |R|$ , i.e.  $den_e(R) = |R|$ . So, if  $den_e(R) < |R|$ , then  $(X, \tau_*)$  is not metrizable<sup>1</sup>.

For the next lemma recall that, for any set  $X \subseteq R$ , a right (respectively left) limit point of  $X$  is a point  $x \in R$  satisfying that, for every  $y > x$  (respectively  $y < x$ ),  $(x, y) \cap X \neq \emptyset$  (respectively  $(y, x) \cap X \neq \emptyset$ ).

**Lemma 6.8.5.** *Suppose that  $den_e(R) < |R|$  and let  $X \subseteq R$  be a subset of cardinality  $|R|$ . Then there exist  $|R|$ -many elements  $x \in X$  that are both a right and left limit points of  $X$ .*

*Proof.* We show that all but at most  $den_e(R)$  many points in  $X$  are right limit points of  $X$  and that the same holds for left limit points. The result then follows from the fact that  $den_e(R) < |R| = |X|$ .

Let  $Y$  be the set of points in  $X$  that are not right limit points of  $X$ . For every  $x \in Y$ , there is some  $x' > x$  such that the interval  $(x, x') = I_x$  is disjoint from  $X$ . The family  $\{I_x : x \in Y\}$  has cardinality  $|Y|$  and contains only non-empty pairwise disjoint intervals. It follows that  $|Y| \leq den_e(R)$ . The proof for the set of left limit points is analogous. □

We may now prove Theorem 6.8.2.

*Proof of Theorem 6.8.2.* Clearly any definably metrizable topological space is metrizable. Fix  $(X, \tau)$ , with  $\dim X \leq 1$ , a definable topological space whose topology is induced by a metric  $d$ . We prove that  $(X, \tau)$  is definably metrizable by describing a definable metric  $\hat{d}$  that induces  $\tau$ . Since every finite metric space is discrete we may assume that  $\dim X = 1$ .

<sup>1</sup>↑For a proof that  $(R, \tau_r)$  is metrizable whenever  $\mathcal{R}$  is a densely ordered countable group see: [math.stackexchange.com/questions/2331814/existence-of-a-certain-near-metric-map-on-an-ordered-divisible-abelian-group](https://math.stackexchange.com/questions/2331814/existence-of-a-certain-near-metric-map-on-an-ordered-divisible-abelian-group)



Let  $D$  be a dense subset of  $R$  of cardinality  $den_e(R)$ . By Remark 2.2.16 we assume that  $X \subseteq R$ .

Consider the definable set  $S = \{x \in X : E_x \setminus \{x\} \neq \emptyset\}$ . We begin by proving the following claim.

**Claim 6.8.6.**  *$S$  is finite.*

Towards a contradiction suppose that  $S$  is infinite. Let  $f : S \rightarrow R_{\pm\infty}$  be the map given by  $x \mapsto \min E_x \setminus \{x\}$ , which, by Lemma 2.2.20, is definable. By Hausdorffness (Lemma 6.2.6 (b)) and o-minimality there exists an interval  $I \subseteq S$  on which  $f$  is  $e$ -continuous and strictly monotonic. Note (see Lemma 6.2.11) that  $I$  is in the  $\tau$ -closure of  $D \cap f(I)$ . Consider the family of  $d$ -balls  $\mathcal{B} = \{B_d(q, \delta) : q \in D \cap f(I), \delta \in (0, \infty) \cap D\}$ . This family has cardinality bounded by  $den_e(R)$  and, by Lemma 6.8.3, contains, for every  $x \in I$ , a subfamily that is a basis of open  $\tau$ -neighborhoods of  $x$ .

Now, by  $T_1$ -ness, for every  $x \in I$  let  $h(x)$  denote a  $\tau$ -neighborhood of  $x$  in  $\mathcal{B}$  such that  $f(x) \notin h(x)$ , i.e.

$$x \in h(x) \subseteq X \setminus \{f(x)\}. \quad (6.4)$$

Since  $|I| = |R|$  and  $|\mathcal{B}| \leq den_e(R)$ , where  $den_e(R) < |R|$ , there must exist, by the pigeonhole principle, some ball  $B \in \mathcal{B}$  such that  $h^{-1}(B)$  has cardinality  $|R|$ . By Lemma 6.8.5 there exists  $x \in h^{-1}(B)$  that is both a right and left limit point of  $h^{-1}(B)$ . Recall that  $f(x) \in E_x$ . Suppose that  $f(x) \in R_x$ . Then, since  $x \in h(x) = B$ , there is some  $z > f(x)$  such that  $(f(x), z) \subseteq B$ . If  $f$  is increasing then, by  $e$ -continuity, there is some  $y > x$  with  $(x, y) \subseteq I$  such that  $f[(x, y)] \subseteq (f(x), z)$ . But then, by (6.4), for every  $x' \in (x, y)$ ,  $h(x') \neq B$ . This contradicts that  $x$  is a right limit point of  $h^{-1}(B)$ . Similarly, if  $f$  is decreasing there is some  $y < x$  with  $(y, x) \subseteq I$  such that  $(y, x) \cap h^{-1}(B) = \emptyset$ , contradicting that  $x$  is a left limit point of  $h^{-1}(B)$ . The argument in the case where  $f(x) \in L_x$  is analogous. This completes the proof of the claim.

We now proceed with the proof of the theorem. By Theorem 6.3.9 and Remark 6.8.4 there exists a partition  $\mathcal{X}$  of  $X$  into finitely many points and intervals where each interval subspace in  $\mathcal{X}$  has the euclidean or discrete topology. Let  $E_S = \cup_{x \in S} E_x$ . By the above claim

and Lemma 2.2.20 both  $S$  and  $E_S$  are finite sets. By passing to a finer partition if necessary we may require that  $\mathcal{X}$  has the following two properties.

- (i) The elements in  $S$  and in  $E_S$  do not belong in any interval in  $\mathcal{X}$ .
- (ii) For any interval  $(a, b) \in \mathcal{X}$  with the discrete subspace topology it holds that, if  $a \in \cup_{x \in X} R_x$ , then  $b \notin \cup_{x \in X} L_x$  and, if  $b \in \cup_{x \in X} L_x$ , then  $a \notin \cup_{x \in X} R_x$  (to see this note that any discrete interval subspace  $I$  that is disjoint from  $E_S$  is also disjoint from  $\cup_{x \in X} E_x$ , and so any proper subinterval of  $I$  has the desired property).

By (i), for any interval  $I = (a, b) \in \mathcal{X}$ ,  $x \in I$  and  $y \in X \setminus I$ , it holds that  $E_x \subseteq \{x\}$  and  $E_y \cap I = \emptyset$ . So, by Lemma 6.2.11,  $I$  is  $\tau$ -open and, if  $y \in \partial_\tau I$ , then it must be that either  $a \in R_y$  or  $b \in L_y$ . In particular, by (ii) and Hausdorffness (Proposition 6.2.8(c)), if  $I$  is discrete then  $|\partial_\tau I| \leq 1$ .

Let  $\mathcal{Y} \subseteq \mathcal{X}$  be the family of all discrete interval subspaces in  $\mathcal{X}$ . Let  $|\mathcal{Y}| = n$ . We prove the theorem by induction on  $n$ .

If  $n = 0$  then  $X$  is cell-wise euclidean. In particular, by Remark 6.1.1, it contains no definable copy of an interval with the discrete or right half-open interval topologies and so, applying Theorem 6.7.1,  $(X, \tau)$  is affine, and in particular it is definably metrizable.

Suppose that  $n > 0$  and let  $\mathcal{Y} = \{I_1, \dots, I_n\}$ . Let  $X' = X \setminus I_n$ . By induction hypothesis the space  $(X', \tau)$  is metrizable with some definable metric  $d'$ . We extend  $d'$  to a definable metric  $\hat{d}$  on  $X$  such that  $\tau|_{\hat{d}} = \tau$ . Let  $I_n = I = (a, b)$ . We consider two cases.

**Case 0:**  $\partial_\tau I = \emptyset$ . This is the case where  $I$  is a clopen subset of  $X$ . Note that the metric  $\min\{1, d'\}$  induces the same topology as  $d'$ , hence by passing to the former if necessary we may assume that  $d' \leq 1$ . We define the metric  $d$  on  $X$  as follows.

- For all  $x, y \in X'$ ,  $\hat{d}(x, y) = d'(x, y)$ .
- For all  $x \in I, y \in X$ ,  $\hat{d}(x, y) = \hat{d}(y, x) = 1$  if  $x \neq y$  and  $\hat{d}(x, y) = \hat{d}(y, x) = 0$  otherwise.

It is easy to check that  $\hat{d}$  is a metric that induces the topology  $\tau$ .

**Case 1:**  $\partial_\tau I \neq \emptyset$ , i.e.  $\partial_\tau I = \{x_0\}$  for some  $x_0 \in X \setminus I$ . We prove the case where  $a \in R_{x_0}$ . The remaining case, where  $b \in L_{x_0}$ , is analogous.

Recall that, by (i) and (ii),  $E_{x_0} \cap (a, b) = \emptyset$  and  $b \notin L_{x_0}$ . Consider the following definable metric  $\hat{d}$  in  $X$ .

- For all  $x, y \in X'$ ,  $\hat{d}(x, y) = d'(x, y)$ .
- For all  $x, y \in I$ ,  $\hat{d}(a, b) = |x - a| + |y - a|$  if  $x \neq y$  and  $\hat{d}(x, y) = 0$  otherwise.
- For all  $x \in I$ ,  $y \in X'$ ,  $\hat{d}(x, y) = \hat{d}(y, x) = |x - a| + d'(y, x_0)$ .

It is routine to check that  $\hat{d}$  is a metric. Clearly it induces the corresponding subspace topologies of  $\tau$  on  $X'$  and  $I$ , and moreover  $I$  is  $d$ -open. Note that an injective definable curve  $\gamma$  in  $I$   $\hat{d}$ -converges if and only if it  $e$ -converges to  $a$  from the right,  $\hat{d}$ -converging to  $x_0$ . By (i) and (ii) the same holds for any  $\tau$ -converging injective definable curve. Hence, by Corollary 6.2.3, we derive that  $\tau_{\hat{d}} = \tau$ . This completes the proof of the theorem.  $\square$

It remains open whether or not Theorem 6.8.2 can be generalized to spaces of dimension greater than one.

## 6.9 An affiness result by Peterzil and Rosel

During the writing of the contents of this chapter the author learned that Peterzil and Rosel were working on similar questions. Their work resulted in [36]. The main theorem (page 1) in said paper is the following affiness result.

**Theorem 6.9.1** ([36], main theorem). *Suppose that  $\mathcal{R}$  expands an ordered field. Let  $(X, \tau)$ , where  $\dim X = 1$ , be a Hausdorff definable topological space. The following are equivalent.*

- (1)  $(X, \tau)$  is affine.
- (2) There is a finite set  $G \subseteq X$  such that the subspace topology  $\tau|_{X \setminus G}$  is coarser than the euclidean topology on  $X \setminus G$ .
- (3) Every definable subset of  $X$  has finitely many definably connected components, with respect to  $\tau$ .
- (4)  $(X, \tau)$  is regular and has finitely many definably connected components.

Their work is in some ways parallel to ours. Their notion of *set of shadows* of  $x$  is effectively the  $e$ -accumulation set of  $x$ . Similarly  $x$  *inhabits the left (respectively right) side of  $y$*  means  $y \in L_x$  (respectively  $y \in R_x$ ). For one-dimensional Hausdorff spaces being *almost coarser than the affine topology* corresponds to being cell-wise euclidean.

In light of Theorem 6.3.9, the implications (1)  $\Leftrightarrow$  (2)  $\Leftrightarrow$  (3) in their theorem are equivalent to our Theorem 6.7.1. To prove it they use a more elementary approach that allows them to note that the result holds with only the assumption that  $\mathcal{R}$  expands an ordered group, when  $X$  is bounded.

To prove (4)  $\Rightarrow$  (1) they use a fact similar to the following, which can be derived from Lemmas 6.4.6 and 6.4.8 (and generalized to all spaces of dimension one using Remark 6.4.9).

**Corollary 6.9.2.** *Let  $(X, \tau)$ ,  $X \subseteq R$ , be a regular Hausdorff definable topological space that is not cell-wise euclidean, has no isolated points, and has no open definable copy of an interval with the  $\tau_r$  or  $\tau_l$  topologies. Then there exists an interval  $I \subseteq R$  and a definable open embedding  $h : (I \times \{0, 1\}, \tau_{lex}) \hookrightarrow (X, \tau)$ . Moreover, for every  $a < b$  in  $I$ , the image by  $h$  of the set  $((a, b] \times \{0\}) \cup ([a, b) \times \{1\})$  is clopen in  $(X, \tau)$ .*

*It follows that every regular Hausdorff one-dimensional definably connected space is cell-wise euclidean.*

They ask if, given a definable topological space  $(X, \tau)$  and  $x \in X$ , the union of all (definable) definably connected sets containing  $x$  is itself definable, i.e. if there exists a (definable) definably connected component containing  $x$ . By means of Theorem 6.4.3 and its generalisation (Remark 6.4.9) one ought to be able to prove that the answer is yes for one-dimensional regular Hausdorff definable topologies, where one such component is either a singleton or an affine space.

They also ask whether or not, for Hausdorff definable topologies of all dimensions, being affine is equivalent to the condition that every definable subset has finitely many definably connected components, and also to being regular and having finitely many definably connected components. The non-equivalence of these three conditions is given by Examples A.14 and A.16.

## 7. GENERAL AFFINENESS RESULTS

### Introduction

Throughout this chapter we work in an o-minimal expansion of an ordered field  $\mathcal{R} = (R, 0, 1, +, \cdot, <, \dots)$ . Recall that, in this setting, any definable curve has domain  $(0, 1)$ , and convergence is understood as  $t$  approaches zero. Recall that we call a definable topological space affine if it is definably homeomorphic to a set with the euclidean topology, and locally affine if every point has an affine neighborhood (Definition 2.2.3). Our aim is to expand on the affineness result for one-dimension spaces (Theorem 6.7.1) in Chapter 6 by studying affineness results for spaces of all dimensions.

In Section 7.1.1 we review the proof of the affineness theorem for regular manifold spaces of van den Dries (Theorem 7.1.2). We make some comments on the limitations of trying to adapt this proof to a more general class of spaces. In Section 7.1.2 we review the proof of the affineness result for definably separable definable metric spaces of Walsberg (Theorem 7.1.5). In Section 7.1.2 we present a new proof of Walsberg’s theorem using the definable Tietze Extension Theorem, and in Section 7.1.2 we present a third more elementary proof. Finally, in Section 7.2 we prove our own affineness result for definably Tychonoff spaces (Theorem 7.2.4).

Throughout this chapter recall that, given a definable metric space  $(X, d)$  and a definable set  $Y \subseteq X$ , we denote by  $d(\cdot, Y)$  the definable distance function given by  $d(x, Y) = \inf\{d(x, y) : y \in Y\}$ . It is easy to see that this function is always continuous. We denote by  $d_e(\cdot, Y)$  the distance function with respect to the euclidean metric.

### 7.1 Affineness results by van den Dries and Walsberg

#### 7.1.1 Van den Dries’ definable manifold spaces

In Chapter 10 of his book [17] van den Dries addresses “definable spaces”, which appeared previously in the o-minimal setting implicitly in [38]. To avoid confusion we refer to these “definable spaces” as “definable manifold spaces”. They are defined to be topological spaces  $(X, \tau)$  that admit a definable atlas of functions  $g_i : U_i \rightarrow R^{n^{(i)}}$ , for  $0 \leq i \leq m$ , where

$X = \bigcup_{0 \leq i \leq m} U_i$ , each  $U_i$  is  $\tau$ -open, and each  $g_i : (U_i, \tau) \rightarrow (R^{n(i)}, \tau_e)$  is a homeomorphism. Moreover, for every  $1 \leq i \leq j \leq m$ , it holds that the transition map  $g_{ij} : g_i(U_i \cap U_j) \rightarrow g_j(U_j \cap U_i)$  given by  $g_{ij}(x) = g_j(g_i^{-1}(x))$  is definable. The topology  $\tau$  is then uniquely determined by the fact that the functions on the atlas are homeomorphisms and each  $U_i$  is open. By passing to a push-forward if necessary, we may redefine these manifold spaces as follows.

**Definition 7.1.1.** *A definable manifold space is a definable topological space that can be covered by a finite family of open definable affine subspaces.*

Van den Dries proved the following.

**Theorem 7.1.2** ([17], Chapter 10, Theorem 1.8). *A definable manifold space is affine if and only if it is regular.*

Before we describe the proof of Theorem 7.1.2 we note in the next corollary how one may use the theorem, together with Corollary 5.4.14, to prove an affineness result for locally affine definable topological spaces in the case where  $\mathcal{R}$  expands  $(\mathbb{R}, <)$ . At the end of Section 7.2 we prove another affineness result (Corollary 7.2.18) for locally affine spaces.

**Corollary 7.1.3.** *Suppose that  $\mathcal{R}$  expands  $(\mathbb{R}, <)$ . Let  $(X, \tau)$  be a locally affine definably compact definable topological space. Then  $(X, \tau)$  is affine.*

*Proof.* By Corollary 5.4.14 if  $(X, \tau)$  is definably compact then it is compact. By topological compactness and local affineness it follows that there are finitely many affine definable  $\tau$ -open subspaces of  $X$  that cover  $X$ . So  $(X, \tau)$  is a definable manifold space, and the result follows from Theorem 7.1.2.  $\square$

Van den Dries proves the “if” direction (the “only if” implication is trivial) of Theorem 7.1.2 constructively, by explicitly describing an appropriate homeomorphism, using ideas previously applied by Robson in the semialgebraic case [44]. Along the way he proves a version of the next crucial lemma (it is a claim in his proof). We include the lemma without proof, and refer the reader to the proof of Theorem 1.8 in [17], Chapter 10. We state it in the language of this thesis. It is worth noting that the lemma can be stated and proved without any assumptions on  $\mathcal{R}$  besides it being an o-minimal expansion of a dense linear order.

**Lemma 7.1.4.** *[[17], Chapter 10, Claim 1 in the proof of Theorem 1.8] Let  $(X, \tau)$  be a regular definable topological space and let  $U \subseteq X$  be a  $\tau$ -open set on which the subspace topology is euclidean. Let  $C$  be the set of points  $x$  in  $\partial_e U$  such that there exists a definable curve in  $U$   $e$ -converging to  $x$  and  $\tau$ -converging outside  $U$ .*

*Then, for every  $y \in U$ , it holds that  $d_e(y, C) > 0$ .*

We now sketch the proof of Theorem 7.1.2.

*Proof of Theorem 7.1.2.* Let  $(X, \tau)$  be a regular definable manifold space. By an inductive argument we may assume that there are just two open sets  $U$  and  $V$  that cover  $X$  and that are affine. Let  $h_1$  and  $h_2$  be definable homeomorphisms from  $U$  and  $V$  respectively into a euclidean space. We may assume, by using the field structure, that the images of  $h_1$  and  $h_2$  are in the same  $R^n$  space, bounded and disjoint. Considering the push-forward of  $(X, \tau)$  by  $h_1 \cup (h_2 \setminus h_1)$ , let  $C_1$  be the set of points  $x$  in  $\partial_e h_1(U)$  such that there is a definable curve  $\gamma$  in  $h_1(U)$   $e$ -converging to  $x$  and converging in the push-forward topology outside  $h_1(U)$ . We define  $C_2$  analogously for  $h_2$  in place of  $h_1$ .

Then, recalling Lemma 7.1.4, define

$$h(x) = \begin{cases} \langle d_e(h_1(x), C_1) & , & d_e(h_1(x), C_1)h_1(x) & , & 0 & , \dots , & 0 \rangle, & \text{if } x \in U \setminus V, \\ \langle d_e(h_1(x), C_1), d_e(h_1(x), C_1)h_1(x), d_e(h_2(x), C_2), d_e(h_2(x), C_2)h_2(x)) \rangle, & \text{if } x \in U \cap V, \\ \langle 0 & , \dots , & 0 & , & d_e(h_2(x), C_2) & , & d_e(h_2(x), C_2)h_2(x) \rangle, & \text{if } x \in V \setminus U, \end{cases}$$

where  $d_e(h_1(x), C_1)h_2(x)$  and  $d_e(h_2(x), C_2)h_2(x)$  refer to scalar multiplication.

Note first that, by Lemma 7.1.4,  $h(x)$  is injective. Continuity is easy to prove by means of definable curves (Corollary 4.5.5), the key observation being that, if  $\gamma$  is a curve in  $U \cap V$  that  $\tau$ -converges to, say, a point in  $V \setminus U$ , then  $h_1 \circ \gamma$   $e$ -converges to some point in  $C_1$ , and so  $d_e(h_1 \circ \gamma, C_1)$  goes to zero.

To prove that  $h$  is a homeomorphism we must show (Corollary 4.5.5) that, if  $h \circ \gamma$  is a curve in the image of  $h$   $e$ -converging to some  $h(x)$ , then  $\gamma$   $\tau$ -converges to  $x$ . Let us fix some  $x \in X$  and definable curve  $\gamma$  in  $X$  such that  $h \circ \gamma$   $e$ -converges to  $h(x)$ . If  $x \in U$  then, by Lemma 7.1.4,  $d_e(h_1(x), C_1) > 0$ . By definition of  $h$  this means that  $\gamma$  has to be (eventually)

in  $U$ . So  $d_e(h_1 \circ \gamma, C_1)$   $e$ -converges to  $d_e(h_1(x), C_1)$ , and  $d_e(h_1 \circ \gamma, C_1)(h_1 \circ \gamma)$   $e$ -converges to  $d_e(h_1(x), C_1)h_1(x)$ . But then

$$e\text{-lim } h_1 \circ \gamma = (e\text{-lim } d_e(h_1 \circ \gamma, C_1)(h_1 \circ \gamma)) \left( e\text{-lim } \frac{1}{d_e(h_1 \circ \gamma, C_1)} \right) = h_1(x).$$

Since  $h_1$  is a homeomorphism it follows that  $\gamma$   $\tau$ -converges to  $x$ . The case where  $x \in V$  is analogous, and this completes the proof.  $\square$

### Comments on the proof of Theorem 7.1.2

The key idea behind the proof of Theorem 7.1.2 lies in defining  $C_1$  and  $C_2$  in such a way that allows both the injectiveness and continuity arguments to work. We now observe how these arguments rely crucially on the fact that  $X$  can be covered by finitely many  $\tau$ -open affine spaces.

Note that a map  $h : X \rightarrow R^n$  is injective if and only if its coordinate maps separate points, i.e. for every distinct pair  $x, y \in X$  there must be a coordinate map  $\pi_i \circ h$  such that  $(\pi_i \circ h)(x) \neq (\pi_i \circ h)(y)$ . Hence, to find a continuous injective map on  $X$  it suffices to find a finite family of continuous  $R$ -valued maps that separates points on  $X$  (and use these as coordinate maps).

Since the set  $U$  in the described proof of Theorem 7.1.2 is affine, the maps  $d_e(h_1(x), C_1)$  and  $d_e(h_1(x), C_1)h_1(x)$  are continuous on  $U$ . By Lemma 7.1.4, they separate any two points in  $U$ . We definably extend these maps to the whole space  $X$  by making them zero outside  $U$  (note that now they also separate any two points  $x \in U$  and  $y \in X \setminus U$ ). By definition of  $C_1$  and the fact that  $U$  is  $\tau$ -open, these extensions are continuous. In particular, to prove continuity (using Corollary 4.5.5) it suffices to check that limits of curves in  $U$   $\tau$ -converging outside  $U$  are maintained. If  $U$  were not  $\tau$ -open then, while the maps  $d_e(h_1 \circ \gamma, C_1)$  and  $d_e(h_1(x), C_1)h_1(x)$  would still be continuous on  $U$ , they would not, by failure of Lemma 7.1.4, necessarily separate points (although if  $U$  were a cell, and therefore locally closed, i.e. open in its closure, then they would) and, perhaps more importantly, their extensions to zero functions outside  $U$  would no longer necessarily be continuous.



We will continue to see in the subsequent subsections how definably continuously extending partial maps on definable topological spaces is a powerful tool in proving affineness results, in particular with regards to piecewise euclidean spaces.

### 7.1.2 Walsberg’s definable metric spaces

Definable metric spaces were studied by Erik Walsberg in [49]. His main result states that such a space is affine if and only if it is definably separable.

**Theorem 7.1.5** ([49], Theorem 7.1). *A definable metric space is affine if and only if it is definably separable.*

He divides the proof of the “if” direction (the “only if” implication follows easily from o-minimality) into two parts. First he shows that every definably separable definable metric space embeds definably into a definably compact one ([49], Proposition 7.19). Then he proves the result for definably compact spaces.

He proves and uses the following two facts.

**Fact 7.1.6** ([49], Lemma 7.15). *Definable metric spaces  $(X, d)$  have the frontier dimension inequality, i.e. for every definable set  $Y \subseteq X$  it holds that  $\dim \partial_a Y < \dim Y$ .*

**Fact 7.1.7** ([49], Corollary 7.17). *Definably separable definable metric spaces are cell-wise euclidean. i.e. they can be partitioned into finitely many cells where the subspace topology is euclidean (Definition 6.7.2).*

Walsberg proves these facts by witnessing each definable metric space  $(X, d)$  as the space of functions  $\{d_x : x \in X\}$  where  $d_x(y) = d(x, y)$ . He studies these families of functions with the uniform and the Hausdorff metrics, using results from [15]. Ultimately, this approach allows him to build the definable compactification of a definably separable definable metric space. We now sketch the second part of his proof, which corresponds to the proof of the following proposition.

**Proposition 7.1.8.** *Every definably compact definable metric space is affine.*

In the next subsections we provide a different proof of Proposition 7.1.8, as well as a more elementary proof of Theorem 7.1.5. In the sketch of Walsberg’s proof of Proposition 7.1.8 that follows we adapt the arguments slightly to avoid the use of the definable Michael’s Selection Theorem of Aschenbrenner and Thamrongtanyalak ([6], Theorem 4.1).

*Proof of Proposition 7.1.8.* Let  $(X, d)$  be a definably compact definable metric space. We apply an inductive argument on the dimension of  $X$ , where the base case  $\dim X = 0$  is trivial. We assume (see Lemma 2.2.16) that  $X$  is bounded. Using a version of Lemma 6.7.4, the question is reduced to the construction of a definably compact euclidean space  $Z'$  and a definable continuous surjective map  $h : (Z', \tau_e) \rightarrow (X, \tau_d)$ . Using Fact 7.1.7, the question is further simplified to the following problem. We fix a cell  $A \subseteq X$  such the  $\tau_d$  subspace topology on  $A$  is euclidean, and must construct a definably compact euclidean space  $Z$  and some continuous definable map  $h : (Z, \tau_e) \rightarrow (X, \tau_d)$  satisfying that  $A \subseteq h(Z)$ .

Let  $C = \partial_d A$ . By the frontier dimension inequality (Fact 7.1.6)  $\dim C < \dim A$ . By induction hypothesis we conclude that  $C$  is affine, and so, by passing to a push-forward of  $(X, \tau)$  if necessary, we assume that the  $\tau_d$  subspace topology on  $C$  is also euclidean.

We now construct a definable function on  $f : A \rightarrow C$  with the following property.

For any definable curve  $\gamma$  on  $A$ , if  $\gamma$   $d$ -converges to some point  $y \in C$ ,  
then  $f \circ \gamma$   $e$ -converges (equivalently  $d$ -converges) to  $y$ . (‡)

The construction of  $f$  is as follows. Recall that every cell is locally closed (open in its closure) in the euclidean topology. By regularity of the euclidean topology, and that fact that the  $\tau_d$  subspace topology on  $A$  is euclidean, it follows that, for every  $x \in A$ , there exists a  $\tau_d$ -neighborhood  $B$  of  $x$  such that  $cl_e(B \cap A) \cap \partial_e A = \emptyset$ . In particular every definable curve in  $B \cap A$   $e$ -converges (equivalently  $d$ -converges) to some point in  $A$ . By definable curve selection (Lemma 4.5.4) we derive that  $B \cap C = \emptyset$ . So  $A$  is also locally closed with respect to  $d$ , i.e.  $d(x, C) > 0$  for every  $x \in A$  (note the similarity between this observation and Lemma 7.1.4). In particular this implies that  $C$  is  $d$ -closed, and so (Proposition 5.4.5 (1)) it is definably compact in the  $\tau_d$  topology (equivalently in the euclidean topology).

By definable choice let  $f$  be any definable function such that, for every  $x \in A$ , it holds that  $d(x, f(x)) < 2d(x, C)$ . Then by the triangle inequality, for every  $y \in C$  and  $x \in A$ ,

$$d(y, f(x)) \leq d(y, x) + d(x, f(x)) < d(y, x) + 2d(x, C).$$

It follows that  $f$  satisfies  $(\ddagger)$ .

The proof is now completed by constructing the continuous definable map  $h : (Z, \tau_e) \rightarrow (X, \tau_d)$  with  $A \subseteq h(Z)$  as follows. Let  $Z$  be the euclidean closure of the graph of  $f$ . Note that, since  $A$  and  $C$  are bounded,  $Z$  is a definably compact euclidean space. Let  $\pi_y(Z)$  denote the projection of  $Z$  to the last coordinates, including the image of  $f$ . Since  $C$  is definably compact in the euclidean topology, it follows that  $\pi_y(Z) \subseteq C$  (in fact by property  $(\ddagger)$  we have that  $\pi_y(Z) = C$  but this is not needed in the proof). Let  $h : Z \rightarrow A \cup C$  be given by

$$h(x, y) = \begin{cases} x & \text{if } x \in A, \\ y & \text{otherwise.} \end{cases}$$

Let  $Z_1$  be the set of points  $\langle x, y \rangle$  in  $Z$  such that  $x \in A = \text{dom}(f)$  and let  $Z_2 = Z \setminus Z_1$ . Then  $h$  is the projection to the first  $x$ -coordinates on  $Z_1$  and to the last  $y$ -coordinates on  $Z_2$ .

Clearly  $A \subseteq h(Z)$ , so it remains to show that the map is continuous. We show that limits of definable curves are maintained (see Corollary 4.5.5). Note first that, since  $A$  is a cell and thus locally closed, the set  $Z_1$  is  $e$ -open and the set  $Z_2$   $e$ -closed in  $Z$ . Let  $\mu$  be a definable curve in  $Z$   $e$ -converging to a point  $\langle x, y \rangle$ . If  $\mu$  is contained in  $Z_2$  then so is  $\langle x, y \rangle$ . In particular  $h \circ \mu$  is a curve in  $C$ , and  $h(x, y) = y \in C$ . Since projections are continuous, it follows that  $h \circ \mu$   $e$ -converges, and equivalently  $d$ -converges, to  $h(x, y) = y$ . Now suppose that  $\mu$  is contained in  $Z_1$ . If  $\langle x, y \rangle$  is in  $Z_1$ , then again the result follows easily from the facts that projections are continuous and the  $\tau_d$  subspace topology in  $A$  is euclidean. Suppose that  $\langle x, y \rangle \in Z_2$ . Note that  $(h \circ \mu)(t)$  is the projection to the first coordinates of  $\mu(t)$ . Let  $\gamma = h \circ \mu$  be the corresponding definable curve in  $A$ . We must show that  $\gamma$   $d$ -converges to  $h(x, y) = y$ .

Since  $Z$  is the euclidean closure of the graph of  $f$  there exists, by definable choice, a definable curve  $\mu'$  in  $Z_1$  of the form  $\mu' = \langle \gamma', f \circ \gamma' \rangle$ , for some definable curve  $\gamma'$  in  $A$ , satisfying  $\|\mu(t) - \mu'(t)\| < t$  for every  $0 < t < 1$ . Since the  $\tau_d$  subspace topology on  $A$  is euclidean we may also ask that  $d(\gamma(t), \gamma'(t)) < t$  for every  $0 < t < 1$ . Clearly  $\mu$  and  $\mu'$   $e$ -converge in  $Z$  to the same point, namely  $\langle x, y \rangle$ . In particular  $f \circ \gamma'$   $e$ -converges to  $y$ . Moreover  $\gamma$  and  $\gamma'$  also  $e$ -converge to the same point  $x \notin A$ . By definable compactness and the fact that the  $\tau_d$  subspace topology on  $A$  is euclidean we have that  $\gamma'$  must  $d$ -converge to some point in  $C$ . Then, by property (‡), it must be that  $\gamma'$   $d$ -converges to  $y$ . Finally, since  $d(\gamma(t), \gamma'(t)) < t$  for all  $0 < t < 1$ , we conclude that  $\gamma$  also  $d$ -converges to  $y$ . This completes the proof.  $\square$

### A proof of Theorem 7.1.5 using the definable Tietze Extension Theorem

A different approach to Theorem 7.1.5, possibly more in line with the general approach taken by van den Dries to prove Theorem 7.1.2, would involve developing ways of definably continuously extending partial functions on definable metric spaces.

Recall that Walsberg proved that any definably separable definable metric space embeds definably into a definably compact one ([49], Proposition 7.19), and so, to prove Theorem 7.1.5, it suffices to do so for definably compact spaces. In particular, the more complicated “if” direction of Theorem 7.1.5 follows from Proposition 7.1.8. Another approach, different from Walsberg’s proof described in the previous section, to the proof of Proposition 7.1.8, is provided by the next lemma.

**Lemma 7.1.9.** *Let  $(X, \tau)$  be a definably compact definable topological space. Suppose that there exists a finite family  $\{f_1, \dots, f_m\}$  of definable continuous functions  $(X, \tau) \rightarrow (R, \tau_e)$  that separate points in  $X$  (i.e. for every distinct pair  $x, y \in X$  there is some  $1 \leq i \leq m$  such that  $f_i(x) \neq f_i(y)$ ). Then the function  $f = \langle f_1, \dots, f_m \rangle : (X, \tau) \rightarrow (R^m, \tau_e)$  is a definable homeomorphism. In particular  $(X, \tau)$  is affine.*

*Proof.* Because the family of continuous functions  $\{f_1, \dots, f_m\}$  separates points the map  $f$  is clearly a continuous injection. Then the result follows from Proposition 5.4.5(4).  $\square$

The Tietze Extension Theorem states that a continuous real-valued function on a closed subset of a metric space can be extended to a continuous function on the whole space. Let us consider a definable version of this result.

**Theorem 7.1.10** (Definable Tietze Extension Theorem). *Let  $(X, d)$  be a definable metric space and  $C \subseteq X$  a closed set. Let  $f : (C, \tau_d) \rightarrow (R, \tau_e)$  be a definable continuous function. There exists a definable continuous function  $g : (X, \tau) \rightarrow (R, \tau_e)$  such that  $g|_C = f|_C$ .*

In its modern form the Tietze Extension Theorem, also referred to as Tietze-Urysohn Extension Theorem, was proved by Uryshon for all normal spaces. The older metric version is due to Tietze. Proofs of the metric version also yield the definable metric version above. One may use for example the following formula due to Riesz (1923, see [5] page 31).

**Lemma 7.1.11.** *Let  $(X, d)$  be an  $\mathcal{R}$ -metric space and  $C \subseteq X$  a closed set. Let  $f : C \rightarrow R$  be a continuous map such that  $f(C) \subseteq (1, 2)$ .*

*Let  $g$  be defined as  $g|_C = f|_C$  and, for any  $x \in X \setminus C$ ,*

$$g(x) = \inf_{y \in C} f(y) \frac{d(x, y)}{d(x, C)}.$$

*Then  $g$  is continuous.*

This approach to the Tietze Extension Theorem is presented in [5] (Lemma 6.6), where Theorem 7.1.10 is proved in definably complete expansions of ordered fields (for sets with the canonical field topology) through an analogue of Lemma 7.1.11. For a detailed proof in the classical metric case, which generalizes to our setting, see [10] (Theorem 4.5.1).

Using the definable Tietze Extension Theorem (Theorem 7.1.10), Lemma 7.1.9 and Facts 7.1.6 and 7.1.7, we may prove Proposition 7.1.8 as follows.

*Proof of Proposition 7.1.8 using Theorem 7.1.10.* We use an inductive argument on  $n = \dim X$ , where the base case  $n = 0$  is trivial. By Remark 2.2.16 we assume that  $X$  is bounded.

By Fact 7.1.7, let  $\mathcal{D}$  be a finite cell partition of  $X$  such that the  $\tau_d$  subspace topology on each cell is euclidean. Let  $A$  be the union of the  $d$ -interiors of the cells in  $\mathcal{D}$  of dimension  $n$ .

Let  $C = X \setminus A$ . By Fact 7.1.6 we have that  $\dim C < \dim X$ . Moreover  $(C, d)$  is definably compact (see Proposition 5.4.5(1)). By induction hypothesis  $(C, d)$  is affine. By Theorem 7.1.10, let  $h_C$  be a definable continuous extension to  $(X, d)$  of a definable homeomorphism from  $(C, d)$  to a euclidean space. Note that  $h_C$  separates points in  $C$ .

Now consider the functions on  $X$  given by  $x \mapsto d(x, C)$  and  $x \mapsto d(x, C)x$  on  $A$ , and by zero on  $X \setminus A$ . Since  $C$  is  $d$ -closed,  $d(x, C) > 0$  for every  $x \in A$ , and so these functions clearly separate any two distinct  $x, y \in A$ , and also any  $x \in A$  and  $y \in X \setminus A$ . Moreover it is easy to prove, using definable curves (Corollary 4.5.5), that these maps are continuous. In particular, in the case of the function given by  $x \mapsto d(x, C)x$  on  $A$ , we use the fact that  $A$  is a finite union of  $d$ -open sets where the  $\tau_d$  subspace topology is euclidean.

We define  $h$  coordinate-wise by taking the coordinate functions of  $h_C$  and the two functions described in the previous paragraph. By Lemma 7.1.9,  $h$  is a homeomorphism onto a euclidean space.  $\square$

Unfortunately, it is not clear whether there exists a definable version of the general Tietze-Urysohn Extension Theorem (or of the simple case given by Uryshon's Lemma), since the arguments in the proofs of those theorems rely on the axiom of choice and do not apparently translate to our setting. We discuss this futher in Remark 7.2.20.

### An elementary proof of Theorem 7.1.5

Inspired by the meticulous construction of van den Dries, we present a proof of Theorem 7.1.5 that follows elementarily from Facts (7.1.6), (7.1.7) and Lemma 7.1.11. In particular, it avoids proving the existence of a definable compactification. It relies heavily on the use of the metric.

*Proof of Theorem 7.1.5.* The “only if” implication is an easy consequence of o-minimality, see for example Remark 3.1.2. Hence we prove that every definably separable definable metric space is affine.

Let  $(X, d)$  be a definable metric space that is definably separable. We proceed by induction on  $n = \dim X$ . The base case  $n = 0$  is trivial. Since  $\mathcal{R}$  expands an ordered field we may assume (see remark 2.2.16) that  $X \subseteq R^n$  and is bounded.

By Fact 7.1.7 let  $\mathcal{D}$  denote a finite cell partition of  $X$  such that the  $\tau_d$  subspace topology on each cell is euclidean. Let  $A = \bigcup\{\text{int}_e(\text{int}_d D) : D \in \mathcal{D}, \dim D = n\}$ . By Fact 7.1.7, and the fact that the euclidean topology has the frontier dimension inequality, we have that  $\dim A = \dim X$  and  $\dim(X \setminus A) < \dim X$ . Moreover note that  $A$  is both  $d$ -open and  $e$ -open.

Let  $C = X \setminus A$ . By inductive hypothesis  $C$  is affine and thus, by passing to a push-forward if necessary, we assume that the  $\tau_d$  topology in  $C$  coincides with the euclidean topology.

Let  $C_0 = C$ ,  $C_1 = \partial_e \partial_e C_0$  and recursively  $C_{i+1} = \partial_e \partial_e C_i$ . Note that it is not necessarily the case that the sets  $C_i$ , for  $i > 0$ , are all empty because, although  $C$  is  $e$ -closed in  $X$ , it is not necessarily  $e$ -closed in  $R^n$ . Note that these sets are nested and, because the  $\tau_d$  subspace topology on  $C$  is euclidean,  $d$ -closed. Moreover by the frontier dimension inequality  $\dim C_{i+1} < \dim C_i$  for every  $i$ , and so there exists some  $m$  such that  $C_m \neq \emptyset$  and  $C_{m+1} = \emptyset$ . Let  $d'_e = \min\{d_e, 1/3\}$ . For any  $0 \leq i < m$  let  $f_i$  denote a definable continuous extension of the map

$$x \mapsto \frac{4}{3} + d'_e(x, \partial_e C_i)$$

on  $C_i$  as described in Lemma 7.1.11. If  $\partial_e C_m \neq \emptyset$  let  $f_m$  be defined in the same way; if  $\partial_e C_m = \emptyset$  we omit this extra function from the definition of our homeomorphism below. By Theorem 7.1.10 let  $f$  be a definable continuous function  $(X, d) \rightarrow R^n$  extending the identity on  $C$ . We define  $h$  as follows:

$$h(x) = \langle d(x, C)x, d(x, C), f_0(x), d(x, C_1), f_1(x), \dots, d(x, C_m), f_m(x), f(x) \rangle,$$

where  $d(x, C)x$  refers to scalar multiplication. We claim that  $h$  is a homeomorphism.

The map is clearly continuous. Moreover note that the functions  $d(x, C)$ ,  $d(x, C)x$  and  $f$  separate points, so  $h$  is injective. It remains to show that the inverse is continuous.

We use Corollary 4.5.5. Let  $h \circ \gamma$  be a definable curve in  $h(X)$   $e$ -converging to a point  $h(x)$ . We must show that  $\gamma$   $d$ -converges to  $x$ . We may assume that  $\gamma$  lies entirely in either  $C$  or  $A$ .

Suppose that  $\gamma$  lies in  $C$ . Then  $d(\gamma(t), C)$  is constantly zero and so the same must be true of  $d(x, C)$ , so, since  $C$  is  $d$ -closed, we have that  $x \in C$ . But then  $f(x) = x$ , and so

$f \circ \gamma = \gamma$   $e$ -converges to  $x$ . Since the  $\tau_d$  subspace topology on  $C$  is euclidean, it follows that  $\gamma$   $d$ -converges to  $x$ .

Suppose now that  $\gamma$  lies in  $A$ . If  $x \in A$  then, because  $A$  is  $d$ -open, we have that  $d(x, C) \neq 0$ . Using the fact that  $h \circ \gamma$   $e$ -converges to  $h(x)$  we may then conclude that

$$e\text{-}\lim \gamma(t) = e\text{-}\lim \frac{d(\gamma(t), C)\gamma(t)}{d(\gamma(t), C)} = \frac{d(x, C)x}{d(x, C)} = x.$$

Since the  $\tau_d$  subspace topology on  $A$  is euclidean it follows that  $\gamma$   $d$ -converges to  $x$ .

Finally, suppose that  $\gamma$  lies in  $A$  and  $x \in C$ . By continuity (Corollary 4.5.5) it suffices to prove that  $\gamma$   $d$ -converges. Let  $i \leq m$  be the largest index such that  $x \in C_i$ . Then  $d(x, C_i) = 0$  and so  $d(\gamma(t), C_i)$  converges to zero. Using definable choice there exists a definable curve  $\mu$  in  $C_i$  such that

$$d(\gamma(t), \mu(t)) < (1+t)d(\gamma(t), C_i) \tag{7.1}$$

for every  $0 < t < 1$ . Note in particular that  $e\text{-}\lim d(\gamma(t), \mu(t)) = 0$ . So, if  $\mu$   $d$ -converges, then so does  $\gamma$  (to the same point) and the proof is complete. From now on we assume that  $\mu$  does not  $d$ -converge. We will reach a contradiction.

Since the  $\tau_d$  subspace topology on  $C_i$  is euclidean, and  $C_i$  is bounded, it must then be that  $\mu$   $e$ -converges to a point in  $\partial_e C_i$ . In particular  $\partial_e C_i \neq \emptyset$ . Note that, since  $x \in C_i$ , if  $d_e(x, \partial_e C_i) = 0$ , then that means that  $x \in C_{i+1}$ , which contradicts the choice of  $i$ . So

$$f_i(x) = \frac{4}{3} + d'_e(x, \partial_e C_i) > 4/3.$$

We now prove that  $e\text{-}\lim f_i \circ \gamma \leq 4/3$  (in fact the limit equals  $4/3$ , but this is not needed). This contradicts that  $f_i \circ \gamma$   $e$ -converges to  $f_i(x)$ .

By Lemma 7.1.11 the map  $f_i$  is defined on  $X \setminus C_i$  as

$$f_i(z) = \inf_{y \in C_i} \left( \frac{4}{3} + d'_e(y, \partial_e C_i) \right) \frac{d(z, y)}{d(z, C_i)}.$$



Recall that  $\mu$   $e$ -converges to a point in  $\partial_e C_i$ , and in particular  $e\text{-}\lim d_e(\mu(t), \partial_e C_i) = 0$ . For any  $\varepsilon > 0$ , if  $0 < t < \varepsilon$  is such that  $d_e(\mu(t), \partial_e C_i) < \varepsilon$ , then, applying (7.1), we have that

$$(f_i \circ \gamma)(t) \leq \left( \frac{4}{3} + d'_e(\mu(t), \partial_e C_i) \right) \frac{d(\gamma(t), \mu(t))}{d(\gamma(t), C_i)} < \left( \frac{4}{3} + \varepsilon \right) (1 + t) < \frac{4}{3} + \frac{7}{3}\varepsilon + \varepsilon^2.$$

This completes the proof. □

## 7.2 An affineness result for definably Tychonoff spaces

In this section we prove an affineness result (Theorem 7.2.4) under the assumption that a definable topological space  $(X, \tau)$  has a sufficiently “rich” definable space of scalars, in the sense that there exists a definable family of continuous functions  $(X, \tau) \rightarrow (R, \tau_e)$  whose weak topology on  $X$  (i.e. the coarsest topology that makes the functions continuous) equals  $\tau$ . Our approach is based on the classical result from functional analysis that a compact Hausdorff topological space  $(X, \tau)$  is metrizable if and only if its space of scalars (continuous real-valued functions on it) with the supremum norm is separable. This approach involves witnessing the space  $(X, \tau)$  as a space of continuous functions (on a separable metric space) with the pointwise convergence topology, and then constructing a metric for said topology. Unfortunately, the classical construction of this metric does not directly translate to the o-minimal definable setting. Hence, we proceed first in a manner similar to the proof of the classical result, using Theorem 7.1.5 to reduce our question to the definable metrizability of the pointwise convergence topology among certain definable families of continuous functions on a euclidean space, and continue by using tools of o-minimality to construct a definable metric. Finally, we use Proposition 7.1.8 from the previous section, which states that definably compact definable metric spaces are affine.

It is worth noting that, classically, by the Tietze-Urysohn Extension Theorem, any compact Hausdorff topological space has a space of scalars whose induced weak topology is the space topology. It is not clear whether there is an analogous fact in the o-minimal field setting. We discuss this in Remark 7.2.20 at the end of the section.

We recall the formal definition of *weak topology*. Let  $X$  be a set, let  $\{Y_u : u \in \Omega\}$  be a family of sets, and let  $\mathcal{F} = \{f_u : X \rightarrow Y_u : u \in \Omega\}$  be a family of functions. Suppose that,

for every  $u \in \Omega$ , the set  $Y_u$  is endowed with a topology  $\tau_u$ . The weak topology  $\tau$  induced on  $X$  by  $\mathcal{F}$  is the coarsest topology that makes all the functions in  $\mathcal{F}$  continuous. If, for each  $u \in \Omega$ ,  $\mathcal{B}_u$  denotes a basis for  $\tau_u$ , then a basis for  $\tau$  is given by finite intersections of sets of the form  $f_u^{-1}(A)$ , for  $u \in \Omega$  and  $A \in \mathcal{B}_u$ .

Given a set  $X$  and a family  $\mathcal{F}$  of functions  $X \rightarrow R$ , by the pointwise convergence topology on  $\mathcal{F}$  we mean the weak topology induced by the projection functions  $f \mapsto f(x)$ , for every  $x \in X$  and  $f \in \mathcal{F}$ , where  $R$  is endowed with the euclidean topology. Even when the family  $\mathcal{F}$  is definable, this topology is not necessarily definable. We present an example to highlight this fact.

**Example 7.2.1.** For any  $t \in (0, 1)$ , let  $f_t : (0, 1) \rightarrow \{0, 1\}$  be the function given by

$$f_t(t) = 1 \text{ and } f_t(s) = 0 \text{ for every } s \in (0, 1) \setminus \{t\}.$$

Let  $\hat{0}$  denote the zero function on  $(0, 1)$ .

The family  $\mathcal{F} = \{f_t : 0 < t < 1\} \cup \{\hat{0}\}$  is definable. One can moreover show that any injective definable curve in  $\mathcal{F}$  converges pointwise to  $\hat{0}$ . We observe that the pointwise convergence topology  $\tau_p$  on  $\mathcal{F}$  is not definable by showing that there does not exist a definable basis of  $\tau_p$ -neighborhoods of  $\hat{0}$ .

Towards a contradiction, let  $\mathcal{B}(\hat{0})$  be a definable basis of  $\tau_p$ -neighborhoods of  $\hat{0}$ . For any  $A \in \mathcal{B}(\hat{0})$ , let  $I_A = \{t \in (0, 1) : f_t \in A\}$ . Note that the family  $\{I_A : A \in \mathcal{B}(\hat{0})\}$  is definable. Now observe that the pointwise convergence topology on  $\mathcal{F}$  is  $T_1$ . It follows that, for every finite family  $\{f_{t_1}, \dots, f_{t_m}\}$  in  $\mathcal{F}$ , there exists some  $A \in \mathcal{B}(\hat{0})$  such that  $t_i \notin I_A$  for every  $1 \leq i \leq m$ . By uniform finiteness and o-minimality it follows that there exists some  $A \in \mathcal{B}(\hat{0})$  and some interval  $J \subseteq (0, 1)$  such that  $I_A \cap J = \emptyset$ . In particular, for any  $s \in J$ , we have that any definable curve in  $\mathcal{F}$  given by  $f_{s+t}$  for all  $t > 0$  small enough does not  $\tau_p$ -converge to  $\hat{0}$ , contradiction.

For the next definition recall that a topological space  $(X, \tau)$  is *completely regular* if, for every  $x \in X$  and closed set  $C \subseteq X$  with  $x \notin C$ , there exists continuous function  $f : X \rightarrow \mathbb{R}$  with  $f(x) = 1$  and  $f|_C = 0$ . One may show that this is equivalent to the condition that the

space of continuous real-valued functions on  $X$  (i.e. scalars) satisfies that its weak topology on  $X$  equals  $\tau$ . Moreover recall that a topological space is *Tychonoff* if it is Hausdorff and completely regular.

**Definition 7.2.2.** *A definable topological space  $(X, \tau)$  is definably completely regular if there exists a definable family  $\mathcal{F}$  of continuous functions  $(X, \tau) \rightarrow (R, \tau_e)$  such that the weak topology induced on  $X$  by  $\mathcal{F}$  equals  $\tau$ .*

*A definable topological space  $(X, \tau)$  is definably Tychonoff if it is Hausdorff and definably completely regular.*

**Remark 7.2.3.** Note that, by model theoretic compactness, our definition of definably completely regular space  $(X, \tau)$  corresponds to the property that, in any elementary extension  $\mathcal{N} = (N, <, \dots)$  of  $\mathcal{R}$ , the interpretation of  $(X, \tau)$  in  $\mathcal{N}$  satisfies that its topology corresponds to the weak topology induced by the space of all  $\mathcal{N}$ -definable continuous  $N$ -valued functions on it.

Note that every definable metric space  $(X, d)$  is definably Tychonoff. In particular the  $\tau_d$  topology is the weak topology induced by the family of functions  $y \mapsto d(x, y)$  for  $x \in X$ .

Our main result of this section is the following.

**Theorem 7.2.4.** *Let  $(X, \tau)$  be a definably compact definable topological space. Suppose it satisfies the following two conditions.*

- (i)  $(X, \tau)$  is definably Tychonoff.
- (ii)  $(X, \tau)$  has the frontier dimension inequality.

*Then  $(X, \tau)$  is affine.*

We prove Theorem 7.2.4 by constructing a definable metric  $d$  for  $(X, \tau)$  such that  $\tau_d = \tau$ , and then applying Proposition 7.1.8. In order to do this we require a number of lemmas. Most of these lemmas deal exclusively with the euclidean topology. Consequently, since there will be no room for confusion, in this section we generally drop the use of the subscript  $e$  in the notation for euclidean closure, frontier, etc, and instead rely heavily on the convention

that the use of any topological notion without explicit reference to a given topology should be understood with respect to the euclidean topology.

The next lemma can be proved using only the assumption that  $\mathcal{R}$  is an o-minimal expansion of an ordered group.

**Lemma 7.2.5.** *Let  $(X, \tau)$  be a Hausdorff definably compact space and let  $\mathcal{F}$  be a definable family of continuous functions  $(X, \tau) \rightarrow (R, \tau_e)$ . If  $(X, \tau)$  has the frontier dimension inequality then  $(\mathcal{F}, \|\cdot\|)$  is definably separable.*

*Proof.* Suppose that  $\mathcal{F}$  is not definably separable. By definable choice there exists a definable family of functions  $\{f_t : t \in I\} \subseteq \mathcal{F}$ , where  $I \subseteq R$  is an interval, and some  $\varepsilon_t > 0$  for every  $t \in I$  such that, for all distinct  $s, t \in I$ , it holds that  $\|f_s - f_t\| > \max\{\varepsilon_s, \varepsilon_t\}$ . By definable choice the values  $\varepsilon_t$  can be chosen definably in  $t \in I$ . By o-minimality (Lemma 2.1.2), after passing to a subfamily of  $\{f_t : t \in I\}$  if necessary, we may fix an  $\varepsilon > 0$  and assume that  $\|f_s - f_t\| > \varepsilon$  for all distinct  $s, t \in I$ . We assume that  $(X, \tau)$  has the frontier dimension inequality and reach a contradiction.

By definable choice, let  $\{x_{s,t} : s, t \in I, s \neq t\}$  be a definable family of points in  $X$  satisfying  $|f_s(x_{s,t}) - f_t(x_{s,t})| > \varepsilon$ . Note that  $x_{s,t} = x_{t,s}$  always. For each  $t \in I$ , let  $x_t^{(0)} = \tau\text{-}\lim_{s \rightarrow t^-} x_{s,t}$  and  $x_t^{(1)} = \tau\text{-}\lim_{s \rightarrow t^+} x_{s,t}$ . By definable compactness the set  $\{x_t^{(0)}, x_t^{(1)} : t \in I\}$  is well defined. It is clearly definable.

**Claim 7.2.6.**  $\tau\text{-}\lim_{t \rightarrow r^-} x_t^{(1)} = \tau\text{-}\lim_{t \rightarrow r^-} x_t^{(0)}$  for every  $r \in I$ .

Let  $x^* = \tau\text{-}\lim_{t \rightarrow r^-} x_t^{(1)}$  and let us fix a definable  $\tau$ -neighborhood  $A$  of  $x^*$ . We show that, for every  $t < r$  sufficiently close to  $r$  it holds that  $x_t^{(0)} \in cl_\tau A$ . Hence  $\tau\text{-}\lim_{t \rightarrow r^-} x_t^{(0)} \in cl_\tau A$ , and the claim follows from Hausdorffness.

For every  $t < r$  sufficiently large  $x_t^{(1)} \in A$ , and so there exists  $\delta_t > 0$  such that the set  $\{x_{s,t} : t < s < t + \delta_t\}$  is contained in  $A$ . By definable choice we may assume that the function  $t \mapsto \delta_t$  is definable. By o-minimality it is continuous on some interval  $(r', r)$ .

We fix  $t \in (r', r)$  and prove that  $x_t^{(0)} \in A$ . Recall that  $x_t^{(0)} = \tau\text{-}\lim_{s \rightarrow t^-} x_{s,t} = \tau\text{-}\lim_{s \rightarrow t^-} x_{t,s}$ . We show that  $x_{t,s} \in A$  for every  $s < t$  close enough to  $t$ .

By continuity of the map  $s \mapsto \delta_s$ , there exist some  $t' < t$  such that, for every  $s \in (t', t)$ , it holds that  $\delta_s > \delta_t/2$ . Let  $s \in (t', t)$  with  $0 < t - s < \delta_t/2$ . Then  $s < t < s + \delta_s$  and, by definition of  $\delta_s$ , we conclude that  $x_{t,s} \in A$ . This completes the proof of the claim.

**Claim 7.2.7.**  $x_t^{(0)} \neq x_t^{(1)}$  for all but finitely many values of  $t \in I$ .

By o-minimality, it suffices to show that, on any subinterval  $J \subseteq I$ , there is  $t \in J$  such that  $x_t^{(0)} \neq x_t^{(1)}$ . For simplicity of notation we take  $J = I$ , but the proof holds on any subinterval. We prove the inequality  $x_t^{(0)} \neq x_t^{(1)}$  for some  $t \in I$  by showing that  $f_t(x_t^{(0)}) \neq f_t(x_t^{(1)})$ .

By o-minimality, after passing to a subinterval of  $I$  if necessary, we may assume that the map  $t \mapsto f_t(x_t^{(1)})$  is  $e$ -continuous. We prove the claim by showing that, after shrinking  $I$  further, it holds that  $|f_s(x_s^{(1)}) - f_t(x_t^{(0)})| > \varepsilon/2$  for all  $s, t \in I$  sufficiently close, and so  $|f_t(x_t^{(1)}) - f_t(x_t^{(0)})| \geq \varepsilon/2$ .

By continuity of the functions in  $\mathcal{F}$  and o-minimality (Lemma 2.1.2) there exists a subinterval  $I'$  of  $I$  and some  $\delta > 0$  such that, for every  $s, t \in I'$ , if  $0 < t - s < \delta$ , then  $|f_t(x_t^{(0)}) - f_t(x_{s,t})| < \varepsilon/4$  and  $|f_s(x_s^{(1)}) - f_s(x_{s,t})| = |f_s(x_s^{(1)}) - f_s(x_{t,s})| < \varepsilon/4$ .

Now let us fix distinct  $s < t$  in  $I'$  with  $|s - t| < \delta$ . Then, by the second triangle inequality,

$$\begin{aligned} |f_s(x_s^{(1)}) - f_t(x_t^{(0)})| &\geq |f_s(x_s^{(1)}) - f_t(x_{s,t})| - |f_t(x_t^{(0)}) - f_t(x_{s,t})| \\ &\geq |f_t(x_{s,t}) - f_s(x_{s,t})| - |f_s(x_s^{(1)}) - f_s(x_{s,t})| - |f_t(x_t^{(0)}) - f_t(x_{s,t})| \\ &> \varepsilon - \frac{\varepsilon}{4} - \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \end{aligned}$$

This proves the claim.

Now consider the definable sets  $X_0 = \{x_t^{(0)} : t \in I\}$  and  $X_1 = \{x_t^{(1)} : t \in I\}$ , which are at most one-dimensional. By the frontier dimension inequality  $\dim cl_\tau(X_0) \leq 1$ , and so, by Corollary 6.7.5 applied to the subspace  $cl_\tau(X_0)$ , the subspace  $X_0$  is cellwise euclidean. Similarly  $X_1$  is cellwise euclidean. Hence, after shrinking  $I$  if necessary, we may assume that the subspace topology on  $X_0$  and on  $X_1$  is the euclidean topology. By o-minimality, after once again shrinking  $I$  if necessary, we may assume that the maps  $t \mapsto x_t^{(0)}$  and  $t \mapsto x_t^{(1)}$  are  $e$ -continuous, and thus continuous as maps  $(I, \tau_e) \rightarrow (X_0, \tau)$  and  $(I, \tau_e) \rightarrow (X_1, \tau)$  respectively. But then, for every  $t \in I$ ,  $\tau\text{-}\lim_{s \rightarrow t^-} x_s^{(0)} = x_t^{(0)}$  and  $\tau\text{-}\lim_{s \rightarrow t^-} x_s^{(1)} = x_t^{(1)}$ . By Claim 7.2.6,

it follows that  $x_t^{(0)} = x_t^{(1)}$  for every  $t \in I$ . This however contradicts Claim 7.2.7, completing the proof of the lemma.  $\square$

In Lemma 7.2.5 the condition of having the frontier dimension inequality is necessary, as shown by the definable Split Interval (Example A.4).

Before we state the next lemma we need to introduce a natural analogue of pseudometric in our setting.

**Definition 7.2.8.** *Let  $X$  be a definable set. An  $(\mathcal{R})$ -pseudometric on  $X$  is a map  $X \times X \rightarrow [0, \infty)$  that is symmetric and satisfies the triangle inequality.*

**Lemma 7.2.9.** *Let  $X$  be a definable set and let  $\mathcal{T} = \{d_t : 0 < t < 1\}$  be a definable family of pseudometrics on  $X$  that separates points, i.e. for every distinct pair  $x, y \in X$  there exists  $0 < t < 1$  such that  $d_t(x, y) \neq 0$ . Suppose that, for every  $x, y \in X$  and  $0 < t \leq t' < 1$ , it holds that  $d_{t'}(x, y) \leq d_t(x, y)$ .*

*Then the weak topology  $\tau$  induced by  $\mathcal{T}$  on  $X$ , i.e. the topology with basis given by sets  $\{y \in X : d_t(x, y) < r \forall t \in F\}$ , for  $x \in X$ ,  $r > 0$  and  $F \subseteq \mathcal{T}$  finite, is definably metrizable.*

*Proof.* By considering the pseudometric  $\min\{d_t, 1\}$  in place of  $d_t$  if necessary (one may check that they generate the same topology  $\tau$ ) we may assume that  $d_t \leq 1$  for every  $0 < t < 1$ .

Let  $d : X \times X \rightarrow \mathcal{R}$  be given by

$$d(x, y) = \sup_{0 < t < 1} td_t(x, y).$$

Since  $\mathcal{T}$  separates points we have that, for every distinct  $x, y \in X$ ,  $d(x, y) \neq 0$ . It is routine to check that  $d$  satisfies the other metric axioms. Since  $\mathcal{T}$  is definable,  $d$  is definable. So  $d$  is a definable metric on  $X$ .

We show that the topology  $\tau_d$  induced by  $d$  on  $X$  equals  $\tau$ . Let  $x \in X$ . For any  $0 < t < 1$  and  $y \in X$  it holds that  $d_t(x, y) \leq \frac{1}{t}d(x, y)$ . In particular, for any finite collection  $0 < t_0 < \dots < t_n < 1$  and any  $r > 0$ , if  $y \in Y$  is such that  $d(x, y) < rt_0$ , then  $d_{t_i}(x, y) < r$  for every  $0 \leq i \leq n$ . Hence  $\tau \subseteq \tau_d$ . Conversely, let us fix  $0 < r < 1$ . Note that, for every  $x, y \in X$  and  $0 < t < 1$ , if  $t < r$  then  $td_t(x, y) \leq t < r$  and, if  $r \leq t$ ,

then  $td_t(x, y) < d_t(x, y) \leq d_r(x, y)$ . Hence, for every  $x, y \in X$  and  $0 < t < 1$ , it holds that  $td_t(x, y) < \max\{r, d_r(x, y)\}$ . So  $d(x, y) \leq \max\{r, d_r(x, y)\}$ . In particular if  $d_r(x, y) \leq r$  then  $d(x, y) \leq r$ . It follows that  $\tau_d \subseteq \tau$ .  $\square$

We now want to prove Lemma 7.2.12, which is a special case of Lemma 7.2.13. It establishes a connection between pointwise and uniform convergence in certain definable families of functions. Before proving it we require a basic o-minimal fact and another lemma.

**Fact 7.2.10.** *Let  $D$  be a definable set and let  $f : D \rightarrow R$  be a definable function. The set  $B$  of points of discontinuity of  $f$  has empty interior in  $D$ .*

This follows from the fact that, by o-minimality, every definable function is continuous in an open subspace of its domain. Hence the set of points of continuity of  $f$  is dense in  $D$ .

**Lemma 7.2.11.** *Let  $A, B \subseteq R^n$  be definable sets with  $B \subseteq cl(A)$ . If  $A \cap B$  has empty interior in  $A$ , then  $A \cap cl(B)$  has empty interior in  $A$ .*

*Proof.* Suppose that  $A \cap cl(B)$  has nonempty interior in  $A$ . Hence there exists an open set  $U \subseteq R^n$  such that  $\emptyset \neq A \cap U \subseteq cl(B)$ .

Let  $\mathcal{D}$  be a finite cell partition of  $B$  and set  $C := \cup_{D \in \mathcal{D}} \partial D$ . Observe that  $\partial B \subseteq C$ . Recall that cells are simply closed, i.e. their frontier is closed. In particular  $C$  is closed. Note that  $cl(B) = B \cup C$ , and so  $A \cap U \subseteq B \cup C$ . So the open set  $U \setminus C$  satisfies that  $A \cap (U \setminus C) \subseteq B$ .

If  $A \cap (U \setminus C) \neq \emptyset$ , then we conclude that  $A \cap B$  has nonempty interior in  $A$ , proving the lemma. Hence onwards we assume that  $A \cap (U \setminus C) = \emptyset$ , meaning that  $A \cap U \subseteq C$ . We will reach a contradiction.

Let  $\mathcal{D}' \subsetneq \mathcal{D}$  be a subfamily of maximal size such that  $A \cap (U \setminus \cup_{D \in \mathcal{D}'} \partial D) \neq \emptyset$ . Set  $C' := \cup_{D \in \mathcal{D}'} \partial D$ . Let us fix  $D \in \mathcal{D} \setminus \mathcal{D}'$ . Note that, by maximality of  $\mathcal{D}'$ , it holds that  $A \cap (U \setminus C') \subseteq \partial D$ . By definition of  $\partial D$ , and since  $U \setminus C'$  is open, there exists some  $x \in D \cap (U \setminus C')$ . In particular there exists  $x \in (U \setminus C') \setminus \partial D$ . Let  $U' = (U \setminus C') \setminus \partial D$ . We have that  $\emptyset \neq D \cap (U \setminus C') \subseteq U'$ . In particular  $D \cap U' \neq \emptyset$ . Since  $U'$  is open,  $D \subseteq B$ , and  $B \subseteq cl(A)$ , we conclude that  $A \cap U' \neq \emptyset$ . This however contradicts the definition of  $U'$ , in particular the fact that  $A \cap (U \setminus C') \subseteq \partial D$ .  $\square$

Note that the following lemma remains true with only the assumption that  $\mathcal{R}$  is an o-minimal expansion of an ordered group. Moreover, we can generalize the lemma to the case where  $X$  is unbounded if we work in the (extended) euclidean topology on  $(R_{\pm\infty})^n$ .

**Lemma 7.2.12.** *Let  $X \subseteq R^n$  be a bounded definable set and let  $\{f_t : 0 < t < 1\}$  be a definable family of functions  $X \rightarrow R$ . Suppose that  $f_t$  converges pointwise to a function  $f$  as  $t \rightarrow 0$ . Then there exists a definable closed set  $Y \subseteq cl(X)$  such that  $Y \cap X$  has empty interior in  $X$  and, for every neighborhood  $U$  of  $Y$ ,  $f_t$  converges uniformly to  $f$  on  $X \setminus U$ .*

*Moreover, let  $\{f_{t,u} : 0 < t < 1, u \in \Omega\}$  be a definable family of maps  $X \rightarrow R$  such that, for every  $u \in \Omega$ ,  $f_{t,u}$  converges pointwise as  $t \rightarrow 0$ . Then the definable sets  $Y_u \subseteq cl(X)$  such that  $\{f_{u,t} : 0 < t < 1\}$  converges uniformly outside any neighborhood  $U$  of  $Y_u$  can be chosen definably over  $u \in \Omega$ .*

*Proof.* We prove the first paragraph of the lemma. The second paragraph follows from the construction of the set  $Y$ .

For each  $x \in X$  and  $\varepsilon > 0$ , let  $t(x, \varepsilon) = \sup\{0 < s < 1 : |(f_t - f)(x)| < \varepsilon \text{ for every } 0 < t < s\}$ . By o-minimality and since  $\{f_t : 0 < t < 1\}$  converges pointwise to  $f$ , this function is well defined and definable. For each  $\varepsilon > 0$  let

$$Y(\varepsilon) = \{y \in R^n : \forall \delta > 0 \exists z \in X \|y - z\| < \delta \text{ and } t(z, \varepsilon) < \delta\}.$$

Let us fix  $\varepsilon > 0$ . Clearly  $Y(\varepsilon) \subseteq cl(X)$ . Note that, since the map  $x \mapsto t(x, \varepsilon)$  is positive, each point in  $Y(\varepsilon)$  is either in  $\partial X$  or is a point of discontinuity of said map. Hence, by Fact 7.2.10,  $Y(\varepsilon) \cap X$  has no interior in  $X$ . Consider  $0 < \varepsilon' < \varepsilon$ . Note that, for any  $x \in X$ ,  $t(x, \varepsilon') \leq t(x, \varepsilon)$ . It follows that  $Y(\varepsilon) \subseteq Y(\varepsilon')$ . Hence, by Lemma 3.2.1,  $Y = \cup_{\varepsilon > 0} Y(\varepsilon)$  satisfies that  $Y \cap X$  has no interior in  $X$ . Applying Lemma 7.2.11, after passing to  $cl(Y)$  if necessary we may assume that  $Y$  is closed.

We now show that, for any neighborhood  $U$  of  $Y$ ,  $f_t$  converges to  $f$  uniformly on  $X \setminus U$ . Let us fix  $U$  and  $\varepsilon > 0$ . We must show that, for all  $t > 0$  small enough,  $|f_t(x) - f(x)| < \varepsilon$  for every  $x \in X \setminus U$ . Suppose this does not hold. Then, by o-minimality, there exists some  $0 < r < 1$  such that, for all  $0 < t < r$ , it holds that  $|f_t(x_t) - f(x_t)| \geq \varepsilon$  for some



$x_t \in X \setminus U$ . Using definable choice we choose  $x_t$  definably in  $t$ . Let  $x_0 = \lim_{t \rightarrow 0} x_t$ . Clearly  $x_0 \in Y(\varepsilon) \subseteq Y$ , contradicting that  $\{x_t : 0 < t < r\} \subseteq X \setminus U$ .  $\square$

We now generalize Lemma 7.2.12 as follows.

**Lemma 7.2.13.** *Let  $X \subseteq \mathbb{R}^n$  be a bounded definable set and let  $\mathcal{F}$  be a definable family of continuous functions  $X \rightarrow \mathbb{R}$  closed under pointwise limits of definable curves. There exists a definable closed set  $Y \subseteq \text{cl}(X)$ , where  $Y \cap X$  has empty interior in  $X$ , such that, for every definable curve  $\gamma$  in  $\mathcal{F}$  and every neighborhood  $U$  of  $Y$ ,  $\gamma$  converges uniformly on  $X \setminus U$ .*

*Proof.* Let

$$Y' = \{x \in X : \exists \varepsilon > 0 \forall t > 0 \exists f \in \mathcal{F} \exists y \in X \|x - y\| < t \text{ and } |f(x) - f(y)| \geq \varepsilon\}.$$

Let  $U$  be a definable neighborhood of  $Y = \text{cl}(Y' \cup \partial X)$  and let  $\gamma = \{f_t : 0 < t < 1\}$  be a definable curve in  $\mathcal{F}$  converging pointwise to some  $f_0 \in \mathcal{F}$ . We first show that  $\gamma$  converges uniformly on  $X \setminus U$ . Then we complete the proof of the lemma by showing that  $\text{cl}(Y' \cup \partial X) \cap X$  has empty interior in  $X$ .

Towards a contradiction suppose that  $\gamma$  does not converge uniformly on  $X \setminus U$ , in which case there exist  $\varepsilon' > 0$  and, for all  $t > 0$  small enough, some  $x_t \in X \setminus U$ , such that  $|f_t(x_t) - f_0(x_t)| \geq \varepsilon'$ . By definable choice the values  $x_t$  can be chosen uniformly in  $t$ . Let  $x_0 = \lim_{t \rightarrow 0} x_t$ . If  $x_0 \in X$  then, by the second triangle inequality, for every  $t$  small enough we have that

$$|f_t(x_t) - f_t(x_0)| \geq |f_t(x_t) - f_0(x_t)| - |f_t(x_0) - f_0(x_t)|$$

and, applying the triangle inequality to  $|f_t(x_0) - f_0(x_t)|$ ,

$$|f_t(x_t) - f_t(x_0)| \geq |f_t(x_t) - f_0(x_t)| - |f_t(x_0) - f_0(x_0)| - |f_0(x_0) - f_0(x_t)|.$$

By continuity of  $f_0$  and the fact that  $\gamma$  converges pointwise to  $f_0$  we conclude that

$$|f_t(x_t) - f_t(x_0)| \geq \frac{\varepsilon'}{2},$$

and consequently  $x_0 \in Y'$ . If, on the other hand,  $x_0 \notin X$ , then  $x_0 \in \partial X$ . Either way this contradicts the fact that  $x_0 \in X \setminus U$ , since  $U$  is a neighborhood of  $Y' \cup \partial X$ . We conclude that  $\gamma$  converges uniformly on  $X \setminus U$ .

By Lemma 7.2.11, if  $Y'$  has empty interior in  $X$ , then the same holds for  $cl(Y' \cup \partial X) \cap X$ . We prove that  $Y'$  has empty interior in  $X$ .

Towards a contradiction suppose that there exists a definable set  $Z \subseteq Y'$  that is open in  $X$ . By definition of  $Y'$ , for every  $x \in Y'$  and  $0 < t < 1$ , let  $\varepsilon_x > 0$ ,  $f_{x,t} \in \mathcal{F}$  and  $y_{x,t} \in X$  be such that  $\|x - y_{x,t}\| < t$  and  $|f_{x,t}(x) - f_{x,t}(y_{x,t})| \geq \varepsilon_x$ . Using definable choice we choose  $f_{x,t}$ ,  $y_{x,t}$  and  $\varepsilon_x$  definably over  $x$  and  $t$ .

For any  $x \in Y'$ , let  $f_{x,0} \in \mathcal{F}$  be the pointwise limit of  $\{f_{x,t} : 0 < t < 1\}$ . Set  $f'_{x,t} := |f_{x,t} - f_{x,0}|$  for each  $0 < t < 1$ . Let  $\mathcal{F}' = \{f'_{x,t} : x \in X, 0 < t < 1\}$ . Note that  $\mathcal{F}'$  is a definable family of continuous nonnegative functions. Moreover, since  $\mathcal{F}$  is closed under pointwise limits of definable curves, the pointwise limit of any definable curve in  $\mathcal{F}'$  is a continuous function. We reach a contradiction by constructing a definable curve in  $\mathcal{F}'$  whose pointwise limit is not continuous, witnessed by a discontinuity at some point in  $Z$ .

Onwards we assume that  $Z$  is an open set. In the case where it is not we may assume, after passing to an open subspace of  $Z$  if necessary, that there exists some projection  $\pi_Z : Z \rightarrow R^{\dim Z}$  that is a homeomorphism onto an open set. Then we may adapt the proof below to the family of functions  $\{f'_{x,t} \circ \pi_Z^{-1} : x \in X, 0 < t < 1\}$  in place of  $\mathcal{F}'$ .

For any  $x \in Y'$ , note that the definable curve of functions  $\{f'_{x,t} : 0 < t < 1\}$  converges pointwise to zero. Moreover, by the triangle inequalities, for every  $x \in Y'$ ,

$$\begin{aligned} f'_{x,t}(y_{x,t}) &= |f_{x,t}(y_{x,t}) - f_{x,0}(y_{x,t})| \\ &\geq |f_{x,t}(y_{x,t}) - f_{x,t}(x)| - |f_{x,0}(y_{x,t}) - f_{x,0}(x)| - |f_{x,0}(x) - f_{x,t}(x)|. \end{aligned}$$

By definition and continuity of  $f_{x,0}$  we may conclude that, for every  $x \in Y'$  and  $t > 0$  small enough,  $f'_{x,t}(y_{x,t}) \geq \varepsilon_x/2$ .

By Lemma 7.2.12, for every  $x \in Y'$  there exists a definable set  $Y'_x$ , where  $Y'_x \cap X$  has no interior in  $X$ , satisfying that, for every neighborhood  $U$  of  $Y'_x$ ,  $\{f'_{x,t} : 0 < t < 1\}$  converges

uniformly on  $X \setminus U$  (to the zero function). Moreover  $Y'_x$  can be chosen definably in  $x \in Y'$ . Note that, for every  $x \in Y'$ ,  $x \in Y'_x$ .

By o-minimality,  $\dim Y'_x < \dim X$  for every  $x \in X$ . Let  $S$  denote the unit sphere in  $R^n$ . Since  $Z$  is open, for every  $x \in Z$  and  $v \in S$ , there exists some  $\delta(x, v) > 0$  such that either  $\{x + tv : 0 < t < \delta(x, v)\} \subseteq Y'_x \cap X$  or  $\{x + tv : 0 < t < \delta(x, v)\} \subseteq X \setminus Y'_x$ . By definable choice we choose  $\delta(x, v)$  definably in  $x$  and  $v$ . Let  $V_x \subseteq S$  be the set of all  $v \in S$  such that  $\{x + tv : 0 < t < \delta(x, v)\} \subseteq X \setminus Y'_x$ .

**Claim 7.2.14.** *For every  $x \in Z$ , the set  $V_x$  is dense in  $S$ .*

If  $V_x$ , for some  $x \in Z$ , is not dense in  $S$ , then by o-minimality (Lemma 2.1.2) there exist a definable set  $S' \subseteq S$  open in  $S$  and some  $\delta' > 0$  such that, for every  $v \in S'$ , the set  $\{x + tv : 0 < t < \delta'\}$  is contained in  $Y'_x \cap X$ . But then this contradicts that  $Y'_x \cap X$  has empty interior. Alternatively, Claim 7.2.14 follows from the good directions Lemma in [17] (Chapter 7, Theorem 4.2).

By Claim 7.2.14 and Lemma 2.1.5 we have that, after passing to an open definable subset of  $Z$  if necessary, there exists a fixed  $v \in S$  such that, for every  $x \in Z$ , it holds that  $v \in V_x$ . By shrinking  $Z$  further if necessary (Lemma 2.1.2), we assume that there exists some  $\delta > 0$  such that, for every  $x \in Z$ ,  $\delta < \delta(x, v)$ . Similarly, we assume that, for every  $x \in Z$ , there exists some  $\varepsilon > 0$  such that  $\varepsilon < \varepsilon_x/2$ . We will reach a contradiction by proving the existence of some  $x \in Z$  and definable curve  $\{y'_{x,t} : 0 < t < 1\}$  in  $X$  that converges to some point  $x' \in \{x + tv : 0 < t < \delta\}$  and satisfies that, for every  $0 < t < 1$ ,  $f'_{x,t}(y'_{x,t}) \geq \varepsilon/2$ . This contradicts the fact that, since  $Y'_x$  is closed and  $\{x + tv : 0 < t < \delta\} \subseteq X \setminus Y'_x$ , there exists, by definition of  $Y'_x$ , some neighborhood  $B$  of  $x'$  in  $X$  such that  $\{f'_{x,t} : t > 0\}$  converges uniformly to the zero function on  $B$ .

For each  $y \in Z$ , let

$$\zeta_y = \inf\{0 < t < \delta/2 : \exists f' \in \mathcal{F}' \text{ such that } f'(y) > \varepsilon \text{ and } \forall s \in (t, t + \delta/2) f'(y + sv) < \varepsilon/2\}$$

if the infimum is not taken over the empty set, and  $\zeta_y = \delta/2$  otherwise.

Suppose that there exists some  $y \in Z$  such that  $\zeta_y = 0$ . Then, by definable choice, there exists a definable curve  $\{g_t : 0 < t < 1\}$  in  $\mathcal{F}'$  satisfying that, for  $t$  small enough,  $g_t(y) > \varepsilon$  and  $g_t(y + sv) < \varepsilon/2$  for every  $s \in (t, t + \delta/2)$ . But then the pointwise limit  $g_0$  of this curve satisfies that  $g_0(y) \geq \varepsilon$  and  $g_0(y + sv) \leq \varepsilon/2$  for every  $s \in (0, \delta/2)$ , contradicting that  $g_0$  is continuous. Hence, onwards we assume that, for every  $y \in Z$ , it holds that  $\zeta_y > 0$ . After passing to an open subset of  $Z$  if necessary (Lemma 2.1.2), we fix  $\zeta > 0$  such that  $\zeta < \zeta_x$  for every  $x \in Z$ .

Let us fix  $x \in Z$ . Recall that, for all  $t > 0$  small enough, it holds that  $f'_{x,t}(y_{x,t}) > \varepsilon_x/2 > \varepsilon$ . Moreover, since  $Z$  is open, for  $t > 0$  small enough we have that  $y_{x,t} \in Z$ , and so, by definition of  $\zeta_{y_{x,t}}$ , there exists some  $s_t \in (\zeta, \zeta + \delta/2)$  such that  $f'_{x,t}(y_{x,t} + s_tv) \geq \varepsilon/2$ . Let  $y'_{x,t} = y_{x,t} + s_tv$ . Let  $s_0 = \lim_{t \rightarrow 0} s_t$ , and let  $x' = x + s_0v$ . Note that  $\zeta \leq s_0 \leq \zeta + \delta/2 < \delta$ , and so  $x' \in X \setminus Y'_x$ . Moreover  $\lim_{t \rightarrow 0} y'_{x,t} = x'$ , and  $f'_{x,t}(y'_{x,t}) \geq \varepsilon/2$  for every  $t > 0$  small enough. Since  $x' \notin Y'_x$  we reach a contradiction. This completes the proof of the lemma.  $\square$

The next lemma is the bulk of the proof of Theorem 7.2.4. It uses Lemmas 7.2.9 and 7.2.13.

**Lemma 7.2.15.** *Let  $X$  be a bounded definable set and let  $\mathcal{F}$  be a definable family of continuous functions  $X \rightarrow R$  closed under pointwise limits of definable curves. Let  $\tau_p$  denote the pointwise convergence topology on  $\mathcal{F}$ . There exists a definably metrizable topology  $\tau$  on  $\mathcal{F}$  that is finer than  $\tau_p$ , and satisfies that, for any definable curve  $\gamma$  in  $\mathcal{F}$ , if  $\gamma$  converges pointwise to some  $f \in \mathcal{F}$ , then it  $\tau$ -converges to  $f$ .*

*Moreover  $\tau$  is maximally coarse among Hausdorff definable topologies, i.e. if  $\mu$  is a definable Hausdorff topology on  $\mathcal{F}$  with  $\mu \subseteq \tau$ , then  $\tau = \mu$ . In particular, if  $\tau_p$  is definable, then  $\tau = \tau_p$ .*

*If  $\mathcal{R}$  expands the ordered field of reals then  $\tau = \tau_p$ .*

*Proof.* We prove the existence of a definable metric  $d$  on  $\mathcal{F}$  whose induced topology  $\tau$  is finer than the pointwise convergence topology  $\tau_p$  and moreover satisfies that, if a definable curve  $\gamma$  converges pointwise to some  $f \in \mathcal{F}$ , then it  $\tau$ -converges to  $f$ . Since  $\mathcal{F}$  is closed under limits of definable curves note that  $\tau$  is definably compact. The fact that  $\tau$  is maximally coarse among Hausdorff definable topologies then follows from Proposition 5.4.5(4), i.e. if  $\mu$

is a Hausdorff definable topology with  $\mu \subseteq \tau$  then the identity map  $id : (\mathcal{F}, \tau) \rightarrow (\mathcal{F}, \mu)$  is a homeomorphism.

We proceed by induction on  $n = \dim X$ . If  $n = 0$  then  $X$  is finite and the pointwise convergence topology equals the uniform convergence topology (given by the supremum norm) and so it is clearly definably metrizable. Suppose that  $n > 0$ .

Let  $Y \subseteq cl(X)$  be as in Lemma 7.2.13. That is,  $Y$  is closed,  $Y \cap X$  has empty interior in  $X$  and, for every neighborhood  $U$  of  $X$ , any definable curve in  $\mathcal{F}$  converges uniformly on  $X \setminus U$ .

Since  $Y \cap X$  has empty interior in  $X$ , we have in particular that  $\dim(X \cap Y) < n$ . If  $X \cap Y = \emptyset$  set  $d_0(f, g) = 0$  for every  $f, g \in \mathcal{F}$ . If  $Y \cap X \neq \emptyset$  then, by induction hypothesis, let  $d_0$  be a definable metric on  $\{f|_{X \cap Y} : f \in \mathcal{F}\}$  that induces a topology as described in the lemma with respect to the pointwise convergence topology. We abuse terminology and write, for any  $f, g \in \mathcal{F}$ ,  $d_0(f, g)$  to mean  $d_1(f|_{X \cap Y}, g|_{X \cap Y})$ , and so  $d_0$  is a definable pseudometric on  $\mathcal{F}$ .

Let  $\mathcal{F}' = \{f|_{X \setminus Y} : f \in \mathcal{F}\}$ . For any  $x \in X$ , recall that  $d_e(x, Y) = \inf\{\|x - y\| : y \in Y\}$ . Note that, since  $Y$  is closed, for every  $x \in X \setminus Y$  it holds that  $d_e(x, Y) > 0$ . For every  $0 < t < 1$  and  $f, g \in \mathcal{F}'$ , let

$$d_t(f, g) = \sup\{|f(x) - g(x)| : x \in X, d_e(x, Y) \geq t\}.$$

In other words,  $d_t$  corresponds to the supremum norm after we restrict the functions on  $\mathcal{F}$  to the outside of a neighborhood of  $Y$  of radius  $t$ .

Note that, for every  $t > 0$ ,  $d_t$  is a definable pseudometric on  $\mathcal{F}'$ . Let  $f, g \in \mathcal{F}'$  be such that  $f \neq g$ , i.e. there exists some  $x \in X \setminus Y$  such that  $f(x) \neq g(x)$ . If  $0 < t < 1$  is such that  $d_e(x, Y) \geq t$ , then  $d_t(f, g) > 0$ . So the family of pseudometrics  $\{d_t : 0 < t < 1\}$  separates points in  $\mathcal{F}'$ . Moreover, clearly  $d_{t'}(f, g) \leq d_t(f, g)$  for every  $f, g \in \mathcal{F}'$  and  $0 < t \leq t' < 1$ . By Lemma 7.2.9, there exists a definable metric  $d_1$  on  $\mathcal{F}'$  whose metric topology is the weak topology induced by the family  $\{d_t : 0 < t < 1\}$ . We abuse terminology and write, for any  $f, g \in \mathcal{F}$ ,  $d_1(f, g)$  to mean  $d_0(f|_{X \setminus Y}, g|_{X \setminus Y})$ , and so  $d_1$  is a definable pseudometric on  $\mathcal{F}$ .

Now, for any  $f, g \in \mathcal{F}$ , let

$$d(f, g) = d_0(f, g) + d_1(f, g).$$

Suppose that  $f(x) \neq g(x)$  for some  $x \in X$ . If  $x \in X \cap Y$  then  $d_0(f, g) \neq 0$ , and if  $x \in X \setminus Y$  then  $d_1(f, g) \neq 0$ . So  $d(f, g) \neq 0$  whenever  $f \neq g$ . It is routine to check that  $d$  satisfies the other metric axioms.

Now we show that the topology  $\tau$  induced by  $d$  is finer than the pointwise convergence topology  $\tau_p$ . Let us fix  $f \in \mathcal{F}$ , some  $\varepsilon > 0$  and a finite nonempty set  $\{x_0, \dots, x_m\} \subseteq X$ . Suppose that the set is indexed in a way that there exists  $0 \leq l \leq m$  such that  $\{x_0, \dots, x_l\} \subseteq X \setminus Y$  and  $\{x_{l+1}, \dots, x_m\} \subseteq X \cap Y$ . By definition of  $d_0$ , there exists some  $\delta > 0$  such that, for every  $g \in \mathcal{F}$ , if  $d_0(f, g) < \delta$ , then  $|f(x_i) - g(x_i)| < \varepsilon$  for every  $l < i \leq m$ . Now let  $0 < t < 1$  be small enough to satisfy that  $d_e(x_i, Y) \geq t$  for every  $0 \leq i \leq l$ . For every  $g \in \mathcal{F}$ , if  $d_t(f, g) < \varepsilon$  then clearly  $|f(x_i) - g(x_i)| < \varepsilon$  for every  $0 \leq i \leq l$ . Finally note that, by definition of  $d_1$ , after shrinking  $\delta$  if necessary it holds that, for every  $g \in \mathcal{F}$ , if  $d_1(f, g) < \delta$  then  $d_t(f, g) < \varepsilon$ . Hence we conclude that, for every  $g \in \mathcal{F}$ , if  $d(f, g) < \delta$ , then  $|f(x_i) - g(x_i)| < \varepsilon$  for every  $0 \leq i \leq m$ . It follows that  $\tau \subseteq \tau_p$ .

Now consider a definable curve  $\{f_t : 0 < t < 1\}$  in  $\mathcal{F}$  that converges pointwise to some  $f_0 \in \mathcal{F}$ . In particular  $\{f_t|_{X \setminus Y} : 0 < t < 1\}$  converges pointwise to  $f|_{X \setminus Y}$ . By Lemma 7.2.13  $d_s(f_t, f_0)$  converges to zero for every  $0 < s < 1$ , and so  $d_1(f_t, f_0)$  converges to zero. Moreover by induction hypothesis  $d_0(f_t, f_0)$  also converges to zero. It follows that  $\{f_t : 0 < t < 1\}$   $\tau$ -converges to  $f_0$ .

It remains to prove only that, if  $\mathcal{R}$  expands the ordered field of reals, then  $\tau = \tau_p$ . Hence let us assume that  $\mathcal{R}$  expands the ordered field of reals. Recall the terminology and basic results of tame pairs (Chapter 2, Section 2.1.4).

Suppose towards a contradiction that  $\tau_p \subsetneq \tau$ . Then there exists some  $f \in \mathcal{F}$  and  $\varepsilon > 0$  such that, for every finite set  $\{x_0, \dots, x_k\} \subseteq X$  and  $\delta > 0$ , there exists some  $g \in \mathcal{F}$  with  $|f(x_i) - g(x_i)| < \delta$  for every  $0 \leq i \leq k$ , satisfying that  $d(f, g) \geq \varepsilon$ . Applying model theoretic compactness there exists an elementary extension  $\mathcal{N}$  of  $\mathcal{R}$ , and some  $g_*$  in  $\mathcal{F}(\mathcal{N})$ , where  $\mathcal{F}(\mathcal{N})$  is the interpretation of  $\mathcal{F}$  in  $\mathcal{N}$ , such that  $|f(x) - g_*(x)| < \delta$  for every  $x \in X$  and

$\delta > 0$  in  $\mathbb{R}$ , and moreover  $d(f, g_*) \geq \varepsilon$  (where we are considering the natural interpretation of  $d$  and  $f$  in  $\mathcal{N}$ ). By the Marker-Steinhorn Theorem (Theorem 2.1.8),  $\mathcal{N}$  is a tame extension. Hence the existence of one such  $g_*$  is a sentence in the language of tame pairs and so, by completeness of the theory of tame pairs, one such function exists in every tame extension. Hence we may assume that  $\mathcal{N} = \mathcal{R}(\xi)$ , a tame extension of  $\mathcal{R}$  by an infinitesimal element. Recall that, in  $\mathcal{R}(\xi)$ , any definable set is definable with parameters only from  $\mathbb{R} \cup \{\xi\}$ , and, if  $\varphi(\xi)$  is a sentence (possibly including parameters from  $\mathbb{R}$ ), then  $\mathcal{R}(\xi) \models \varphi(\xi)$  if and only if  $\mathcal{R} \models \varphi(t)$  for all  $0 < t < s$  in  $\mathbb{R}$  for some  $s > 0$ . In particular, let  $\phi(x, z, \xi)$  be a sentence that defines  $g_*(x) = z$  and, for any  $0 < t < s$  in  $\mathbb{R}$ , where  $s > 0$  is small enough, let  $g_t$  be the function in  $\mathcal{F}$  defined by  $\phi(x, z, t)$ . Then note that, by properties of  $g_*$ , the definable curve  $\{g_t : 0 < t < s\}$  in  $\mathcal{F}$  converges pointwise to  $f$ , and moreover satisfies that, for every  $t$  small enough,  $d(f, g_t) \geq \varepsilon$ . This however contradicts that pointwise limits of definable curves in  $\mathcal{F}$  coincide with  $\tau$ -limits. This completes the proof of the lemma.  $\square$

The following is an example where Lemma 7.2.15 applies but  $\tau \neq \tau_p$ .

**Example 7.2.16.** Suppose that  $\mathcal{R}$  contains an infinitesimal element  $\varepsilon$ . Consider, for every  $\varepsilon \leq u \leq 1 - \varepsilon$  the following function defined on the closed interval  $[0, 1]$ .

$$f_u(x) = \begin{cases} 1 - \frac{|x - u|}{\varepsilon} & \text{for } |x - u| < \varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

In particular  $f_u$  is a continuous function that is zero outside  $(u - \varepsilon, u + \varepsilon)$  and satisfies  $f_u(u) = 1$ . Let  $\mathcal{F} = \{f_u : \varepsilon \leq u \leq 1 - \varepsilon\} \cup \{\mathbf{0}\}$ , where  $\mathbf{0}$  denotes the zero function. One may show, using arguments analogous to the ones in Example 7.2.1, that the pointwise topology on  $\mathcal{F}$  is not definable.

We may now prove Theorem 7.2.4.

*Proof of Theorem 7.2.4.* Let  $(X, \tau)$  be a definably compact definably Tychonoff definable topological space with the frontier dimension inequality. We prove that  $(X, \tau)$  is definably metrizable. The affineness then follows from Proposition 7.1.8.

Since  $(X, \tau)$  is definably Tychonoff there exists a definable family  $\mathcal{F}$  of continuous functions  $(X, \tau) \rightarrow (R, \tau_e)$  whose induced weak topology on  $X$  is  $\tau$ . As usual we implicitly identify every function in  $\mathcal{F}$  with its respective index parameters. By definable compactness and Lemma 5.4.6 the functions in  $\mathcal{F}$  are bounded and reach their maximum and minimum. In particular  $(\mathcal{F}, \|\cdot\|)$  is a definable metric space. We interpret  $X$  as a space of continuous functions on  $(\mathcal{F}, \|\cdot\|)$  given by  $x(f) \mapsto f(x)$  for  $x \in X$  and  $f \in \mathcal{F}$ . Note that, under this interpretation, the pointwise convergence topology on  $X$  is precisely  $\tau$ .

Since  $(X, \tau)$  is Hausdorff, definably compact and has the frontier dimension inequality, by Lemma 7.2.5 the definable metric space  $(\mathcal{F}, \|\cdot\|)$  is definably separable. By Theorem 7.1.5 it is affine. Let  $Z$  be a definable set and  $g$  a definable homeomorphism  $(\mathcal{F}, \|\cdot\|) \rightarrow (Z, \tau_e)$ . By considering if necessary, for  $x \in X$ , the function  $x \circ g^{-1} : Z \rightarrow R$  in place of  $x$ , we interpret  $(X, \tau)$  to be a definable family of continuous functions on the euclidean space  $(Z, \tau_e)$ , where  $\tau$  is the pointwise convergence topology.

By Remark 2.2.16 we assume that  $Z$  is bounded. Recall that  $(X, \tau)$  is definably compact (in particular by Proposition 5.4.17 it is definably curve-compact). Applying Lemma 7.2.15 we conclude that  $\tau$  is definably metrizable. This completes the proof of the theorem.  $\square$

The following is an immediate consequence of Theorem 7.2.4, by noting that the pointwise convergence topology among a definable family of functions is always, by definition, definably Tychonoff, and recalling the fact that definable compactness and definable curve-compactness are equivalent notions in  $\mathcal{R}$  (Proposition 5.4.17).

**Corollary 7.2.17.** *Let  $X$  be a definable set and let  $\mathcal{F}$  be a definable family of functions  $X \rightarrow R$  closed under pointwise limits of definable curves. Suppose that the pointwise convergence topology  $\tau_p$  on  $\mathcal{F}$  is definable. If  $(\mathcal{F}, \tau_p)$  has the frontier dimension inequality then it is affine.*

Theorem 7.2.4 applies to locally affine spaces as follows.

**Corollary 7.2.18.** *Let  $(X, \tau)$  be a definably compact Hausdorff definable topological space with the frontier dimension inequality. If  $(X, \tau)$  is locally affine then it is affine.*

*Proof.* Let  $(X, \tau)$  be a definably compact Hausdorff definable topological space with the frontier dimension inequality. Suppose that it is locally affine. By Theorem 7.2.4 it suffices



to prove that  $(X, \tau)$  is definably completely regular. We prove that, for any  $x \in X$  and a  $\tau$ -neighborhood  $U$  of  $x$ , there exists a continuous definable function  $f_{x,U} : (X, \tau) \rightarrow (R, \tau_e)$  satisfying that  $f_{x,U}(x) > 0$  and  $f_{x,U}(y) = 0$  for every  $y \in X \setminus U$ . Let  $\mathcal{B}$  be a definable basis for  $\tau$ . By the usual model theoretic compactness argument it follows that  $f_{x,U}$  as described can be chosen definably in  $x \in X$  and  $U \in \mathcal{B}$  with  $x \in U$ . Let  $\mathcal{F}$  be the corresponding definable family of functions. It is easy to see that the weak topology induced by  $\mathcal{F}$  on  $X$  is  $\tau$ .

Hence let us fix  $x \in X$  and a  $\tau$ -neighborhood  $U$  of  $x$ . Since  $(X, \tau)$  is locally affine, by passing to a smaller  $\tau$ -neighborhood if necessary we may assume that there exists a definable homeomorphism  $h : (U, \tau) \rightarrow (R^n, \tau_e)$ . By regularity of  $\tau$  (Lemma 5.4.7) let  $V$  be a  $\tau$ -open  $\tau$ -neighborhood of  $x$  such that  $cl_\tau(V) \subseteq U$ . Then, by Proposition 5.4.5 (1) and (3),  $C = h(\partial_\tau V)$  is a closed and bounded set. Let  $d_e$  denote the euclidean metric on the image of  $h$ . Consider the definable function  $f$  on  $X$  given by  $f(y) = 0$  if  $y \in X \setminus V$  and  $f(y) = d_e(h(y), C)$  otherwise. Using definable curves (Corollary 4.5.5) one easily checks that this map is continuous. Since  $C$  is closed it holds that  $f(x) > 0$ . Moreover, clearly  $f(y) = 0$  for every  $y \in X \setminus U$ . This completes the proof.  $\square$

It is an open question whether the assumption of having the frontier dimension inequality is necessary in the above corollary. It seems quite likely that any definably compact Hausdorff locally affine space already satisfies this condition.

We may formulate Theorem 7.2.4 in such a way that it resembles more closely the classical metrization result in functional analysis as follows.

**Corollary 7.2.19.** *Let  $(X, \tau)$  be a definably Tychonoff definable topological space, and let  $\mathcal{F}$  denote a definable family of functions  $(X, \tau) \rightarrow (R, \tau_e)$  such that their induced weak topology on  $X$  equals  $\tau$ . Suppose that  $(X, \tau)$  is definably compact. Then the following are equivalent.*

- (i)  $(X, \tau)$  is definably metrizable.
- (ii)  $(\mathcal{F}, \|\cdot\|)$  is definably separable.

*Proof.* Lemma 7.5 in [49] states that definable metric spaces have the frontier dimension inequality. Consequently the implication (i)  $\Rightarrow$  (ii) follows from Lemma 7.2.5.

The implication (ii)  $\Rightarrow$  (i) follows from the proof of Theorem 7.2.4. In particular in said proof we apply Lemma 7.2.5 to show that the space  $(\mathcal{F}, \|\cdot\|)$  is definably separable, and then explain how can one apply Lemma 7.2.15 to prove the definable metrizability of  $(X, \tau)$ .  $\square$

**Remark 7.2.20.** In general topology, results such as Urysohn's Lemma or, equivalently, the Tietze-Urysohn Extension Theorem, ensure that Hausdorff normal topological spaces (i.e. Hausdorff spaces where any two disjoint closed sets can be separated by neighborhoods) are completely regular. In particular this applies to Hausdorff compact spaces, since they are always normal. It is not clear whether or not these results have definable analogues in the o-minimal setting. In particular all the proofs that were found of Urysohn's Lemma and the Tietze-Urysohn Extension Theorem by the author make use of the axiom of choice/ Zorn's Lemma.

We end this section with some open questions.

**Questions 7.2.21.** *Let  $(X, \tau)$  be a definable topological space.*

- (i) *If  $(X, \tau)$  is regular and has the frontier dimension inequality, is it definably completely regular?*
- (ii) *If  $(X, \tau)$  is regular, is it definably completely regular?*
- (iii) *If  $(X, \tau)$  is definably compact and Hausdorff, is it definably Tychonoff?*

Since every definably compact Hausdorff space is regular (Lemma 5.4.7), a positive answer to the second question would also answer the other two. By Theorem 7.2.4, a positive answer to the first question would imply that a definably compact Hausdorff space is affine if and only if it has the frontier dimension inequality.

## A. EXAMPLES

In this appendix we compile examples that witness the heterogeneity of definable topological spaces with reference to their (definable) topological properties. They help frame the results in this thesis, in particular in Chapter 6, and their limitations when trying to improve or generalize them.

The examples are given in the language  $(0, 1, +, -, \cdot, <)$ , where  $\mathcal{M}$  is assumed to expand an ordered group  $(M, 0, +, -, <)$  or an ordered field  $(M, 0, 1, +, -, \cdot, <)$  whenever the corresponding function symbols are involved. Throughout we fix infinitely many parameters  $0 < 1 < 2 < \dots$  in  $M$ , in such a way that it will be clear from context when these numerals denote elements of  $M$  and when they are just natural numbers. Whenever we assume that  $\mathcal{M}$  expands an ordered group or field these parameters have their natural interpretations.

Since we are working in the generality of an o-minimal structure, it is important to note that we will not address certain topological properties of definable topological spaces, because they are dependent on the specifics of the underlying structure  $\mathcal{M}$ . These include compactness, connectedness, separability, normality or metrizability. We focus instead on their definable analogues. All the examples that are generalizations of classical topological spaces (e.g. definable Split Interval, definable Alexandrov Double Circle...) behave, in terms of their definable topological properties, exactly like their classical counterparts, the only exception to this being normality (see Example A.12).

**Example A.1** (Euclidean topology). The euclidean topology  $\tau_e$  on  $M^n$  has definable basis

$$\left\{ \prod_{1 \leq i \leq n} (x_i, y_i) : x_i < y_i, 1 \leq i \leq n \right\}.$$

It is  $T_3$ , definably separable, definably connected and definably metrizable. It is moreover definably compact when restricted to a closed and bounded set.

**Example A.2** (Discrete topology). The discrete topology  $\tau_s$  on  $M^n$  has definable basis

$$\{\{x\} : x \in M^n\}.$$

Note that this topology is definable on any definable set in any model-theoretic structure. It is  $T_3$  and definably metrizable.

**Example A.3** (Half-open interval topologies). The right half-open interval topology (or lower limit topology)  $\tau_r$  has definable basis

$$[x, y) \text{ for } x, y \in M, x < y.$$

The space  $(\mathbb{R}, \tau_r)$  is classically called the Sorgenfrey Line.

The left half-open interval topology (or upper limit topology)  $\tau_l$  has definable basis

$$(x, y] \text{ for } x, y \in M, x < y.$$

These topologies are  $T_3$  and definably separable. They are also totally definably disconnected (the only definably connected subspaces are singletons) and not definably metrizable (see Proposition 2.2.8).

**Example A.4** (Definable Alexandrov Double Arrow or definable Split Interval). Let  $X = [0, 1] \times \{0, 1\}$ . The definable Alexandrov Double Arrow (or definable Split Interval) is the space  $(X, \tau_{lex})$ , where  $\tau_{lex}$  denotes the topology induced by the lexicographic order on  $X$ . It is  $T_3$ , definably compact, definably separable and totally definably disconnected.

It is not definably metrizable since the bottom line  $[0, 1] \times \{0\}$  is definably homeomorphic to  $([0, 1], \tau_l)$  and the top line  $[0, 1] \times \{1\}$  to  $([0, 1], \tau_r)$ . It is also worth noting that  $(X, \tau)$  does not satisfy the f.d.i., since  $\partial([0, 1] \times \{0\}) = [0, 1] \times \{1\}$ . Moreover, one may show that  $[0, 1] \times \{0\}$  is not a boolean combination of open definable sets, which was a tameness condition for definable topologies considered by Pillay in [38].

The following example, the definable Alexandrov  $n$ -line, which was already introduced in Example 6.4.2 and which plays a crucial role in Theorem 6.4.3, was motivated by the Alexandrov Double Circle.

**Example A.5** (Definable Alexandrov  $n$ -line). For any  $n > 0$ , let  $X_n = M \times \{0, \dots, n-1\}$ . For any  $y < x < z$  in  $M$ , let  $A(x, y, z) = \{(x, 0)\} \cup ((y, z) \setminus \{x\}) \times M$ . Let  $\tau_{Alex}$  be the

topology on  $M^2$  with definable basis  $\{A(x, y, z) : y < x < z\} \cup \{\{x, y\} : y \neq 0\}$ . The definable Alexandrov  $n$ -line is the definable topological space  $(X_n, \tau_{Alex})$ .

This space is Hausdorff and regular. If  $n = 1$  then it is simply a euclidean space. Suppose that  $n > 1$ . The subset  $M \times \{i\}$  for any  $i > 0$  contains only isolated points, so the space is not definably separable. The subspace  $[0, 1] \times \{0, \dots, n - 1\}$  is definably compact but not definably separable, so not definably metrizable. Hence  $(X_n, \tau_{Alex})$  for  $n > 1$  is not definably metrizable.

Moreover, for any  $0 < n < m < \omega$ , the spaces  $(X_n, \tau_{Alex})$  and  $(X_m, \tau_{Alex})$  are not definably homeomorphic. In fact they are not even in definable bijection, since they have different Euler characteristic (see Chapter 4 in [17]). Specifically every finite cell partition of  $X_n$  will contain  $n$  more cells of dimension 1 than points. Meanwhile every cell partition of  $X_m$  will contain  $m$  more cells of dimension 1 than points. By o-minimal cell decomposition every definable injection  $X_n \rightarrow X_m$  can be decomposed into disjoint definable bijections between singletons and between 1 dimensional cells.

**Example A.6.** Consider the following definable basis for a topology on  $M$ .

$$\{(-\infty, x] : x \in M\}.$$

The resulting space is  $T_0$  but not  $T_1$ . Any subspace with more than one element fails to be  $T_1$ ; in particular no interval subspace of this space has the euclidean, discrete or half-open interval topologies. Consequently, the  $T_1$  assumption in Proposition 6.3.1 cannot be weakened to  $T_0$ .

**Example A.7.** Consider the definable family of sets

$$\{(-\infty, x) \cup (y, z) : x < y < z\}.$$

It is a basis for a topology on  $M$  that is  $T_1$  but not Hausdorff.

Any finite definable partition of  $M$  must include an interval of the form  $(-\infty, x)$ , whose subspace topology is not Hausdorff. In particular  $(X, \tau)$  cannot be decomposed into finitely

many definable subspaces with the euclidean, discrete or half-open interval topologies. It follows that the Hausdorffness assumption in Theorem 6.3.9 cannot be weakened to  $T_1$ -ness.

**Example A.8.** Let  $X = [0, 1) \cup \{2\}$  and consider a topology  $\tau$  on  $X$  such that the subspace topology  $\tau|_{[0,1)}$  is euclidean and a basis of open neighborhoods of  $\{2\}$  is given by

$$\{(0, x) \cup \{2\} : 0 < x < 1\}.$$

This topology is clearly definable and  $T_1$  but not Hausdorff, since points 0 and 2 fail to have disjoint neighborhoods. In particular, it is not regular. It is easy to observe that it has the frontier dimension inequality (f.d.i.). Since it fails to be regular, it illustrates the necessity of the Hausdorffness assumption in Proposition 6.2.14. Moreover we note, by considering the partition into subspaces  $(0, 1)$  and  $\{2\}$ , that it is cell-wise euclidean. So it is not true that every cell-wise euclidean one-dimensional space is Hausdorff, and in particular affine.

**Example A.9.** Let  $X = M \times \{0, 1\}$  and consider a topology  $\tau$  on  $X$  given by the basis

$$\{\langle x, 0 \rangle\} \cup ((x, y) \times \{1\}) : x < y \cup \{(z, x) \times \{1\} : z < x\}.$$

This space is Hausdorff but not regular, since  $M \times \{0\}$  is a closed set and, for any  $x \in M$  and any neighborhood  $U = (z, x) \times \{1\}$  of  $\langle x, 1 \rangle$ ,  $cl_\tau U \cap (M \times \{0\}) = [z, x) \times \{0\} \neq \emptyset$ . Moreover note that, since  $\partial_\tau(M \times \{1\}) = M \times \{0\}$ , it does not have the f.d.i.

Recall that any definable subspace of a definably separable definable metric space is also definably separable (see Proposition 3.1.6). Example A.9 is definably separable, but the subspace  $M \times \{0\}$  is infinite and discrete, showing that definable separability, much like separability, is not in general a hereditary property.

**Example A.10.** Let  $X = \langle 0, 0 \rangle \cup [0, 1) \times (0, 1)$ . Consider the topological space  $(X, \tau)$ , where the subspace  $X \setminus \langle 0, 0 \rangle$  is euclidean, and a basis of open neighborhoods for  $\langle 0, 0 \rangle$  is given by sets

$$A(t) = \langle 0, 0 \rangle \cup ((0, 1) \times (0, t))$$

for  $0 < t < 1$ . The topology  $\tau$  is clearly definable and Hausdorff. Moreover, for any  $0 < t < 1$ , the  $\tau$ -closure of  $A(t)$  is  $\{\langle 0, 0 \rangle\} \cup [0, 1) \times (0, t]$ , and so the space is not regular, since the point  $\langle 0, 0 \rangle$  and the closed set  $\{0\} \times (0, 1)$  are not separated by neighborhoods.

Since  $(X, \tau)$  is  $T_1$  and the subspace  $X \setminus \{\langle 0, 0 \rangle\}$  is euclidean it easily follows that  $(X, \tau)$  satisfies the f.d.i. Hence  $(X, \tau)$  is Hausdorff and has the f.d.i but fails to be regular, serving as a counterexample to the generalization of Proposition 6.2.14 to spaces of dimension greater than one.

Moreover, the space can be partitioned into two euclidean subspaces, namely  $\{\langle 0, 0 \rangle\}$  and  $X \setminus \{\langle 0, 0 \rangle\}$ ; in particular it contains no definable copy of an interval with either the discrete or the right half-open interval topology. However, it is not metrizable, since it is not regular. Hence it is a counterexample to a generalization of Theorem 6.7.1 to spaces of dimension two.

**Example A.11** (Example 5.4.20). Suppose that  $\mathcal{M} = (M, <)$ . Then the following is an example of a non-Hausdorff space that is definably curve-compact but not definably compact. For a proof of this we refer the reader to Example 5.4.20.

Let  $X = \{\langle x, y \rangle \in M^2 : y < x\}$ . Consider the family  $\mathcal{B}$  of sets

$$A(x', x'', x''', y', y'', y''') = \{\langle x, y \rangle : y < y', y < x\} \cup \\ \{\langle x, y \rangle : y'' < y < y''' \wedge (y < x < y''' \vee x' < x < x'' \vee x''' < x)\}$$

definable uniformly over  $y' < y'' < y''' < x' < x'' < x'''$ .

The family  $\mathcal{B}$  is a definable basis for a topology  $\tilde{\tau}$ . This topology is  $T_1$  but not Hausdorff. It is also definably curve-compact but not definably compact (see Example 5.4.20). It follows that the connection between these two notions of compactness described in Theorem 5.4.9 is tight.

For the subsequent Examples A.12, A.13, A.14 and A.16, let  $B_2(\langle x, y \rangle, t)$ , for  $\langle x, y \rangle \in M^2$  and  $t > 0$ , denote the ball in the 2-norm of center  $\langle x, y \rangle$  and radius  $t$ , namely

$$B_2(\langle x, y \rangle, t) = \{\langle x', y' \rangle \in M^2 : (x - x')^2 + (y - y')^2 < t^2\}.$$

**Example A.12** (Definable Moore Plane). Let  $X = \{\langle x, y \rangle \in M^2 : y \geq 0\}$  be the closed upper half-plane. Let  $\mathcal{B}_e$  be a definable basis for the euclidean topology in  $\{\langle x, y \rangle \in M^2 : y > 0\}$  and, for any  $x \in M$ , let

$$A(x, \varepsilon) = B_2(\langle x, \varepsilon \rangle, \varepsilon) \cup \{\langle x, 0 \rangle\}.$$

The family  $\mathcal{B} = \mathcal{B}_e \cup \{A(x, t) : x \in M, t > 0\}$  is clearly definable and forms a basis for a topology  $\tau$ . We call the space  $(X, \tau)$  the definable Moore Plane.

This space is  $T_3$  and definably separable but not definably metrizable since the subspace  $M \times \{0\}$  is infinite and discrete (see Lemma 3.1.6). When  $\mathcal{M}$  expands the field of reals the Moore Plane is a classical example of a separable non-normal space (in particular not metrizable).

It is worth noting that, even though our definition of definable normality (Definition 6.2.15) seems the natural adaptation of the classical notion, the classical Moore Plane fails to be normal. Meanwhile one may show that the definable Moore Plane is definably normal. This suggests that our notion of definable normality might not be adequate.

**Example A.13** (Definable Alexandrov Double Circle). Let  $X = C_1 \cup C_2$ , where  $C_1$  and  $C_2$  denote respectively the unit circle and circle of radius two in  $M^2$  centered at the origin. Let  $f : C_1 \rightarrow C_2$  be the natural  $e$ -homeomorphism given by  $x \mapsto 2x$ . Let  $\mathcal{B}_2 = \{\{x\} : x \in C_2\}$  and  $\mathcal{B}_1 = \{(B_2(x, t) \cap C_1) \cup f(B_2(x, t) \cap C_1 \setminus \{x\}) : x \in C_1, t > 0\}$ . The definable Alexandrov Double Circle is the topology on  $X$  generated by the basis  $\mathcal{B}_1 \cup \mathcal{B}_2$ .

This space is definably compact and Hausdorff, but not definably separable, since  $C_2$  is an infinite definable set of isolated points. It follows (see Lemma 7.4 in [49], which states that any definably compact definable metric space is definably separable) that it is not definably metrizable. It also fails to have the f.d.i., since the outer circle  $C_2$  is a dense subset.

When  $\mathcal{M}$  expands the field of reals this space is simply called the Alexandrov Double Circle and is a classical example of a compact non-separable space (hence not metrizable).

The following example shows that there exists a Hausdorff two-dimensional definable topology that does not contain a definable copy of an interval with the discrete or lower



limit topology but still fails to be cell-wise euclidean. In particular the space is not affine but, by Theorem 6.7.1, any one-dimensional subspace is affine (it is “line-wise” affine). This is proved in Proposition A.15.

**Example A.14** (The definable hollow plane). We construct a basis for a topology  $\hat{\tau}$  on  $M^2$  by considering, for each point  $x$ , a basis of open neighborhoods given by open euclidean balls minus the graph of  $s = t^2$  for  $0 < t$  translated to have its origin at  $x$ . That is, for a given  $x = \langle x_1, x_2 \rangle \in M^2$ , let  $\Gamma_x := \{\langle x_1 + t, x_2 + t^2 \rangle : t > 0\}$ . Now let  $\mathcal{B}$  be given by sets:

$$A(x, t) := B_2(x, t) \setminus \Gamma_x,$$

for  $x \in M^2$  and  $t > 0$ . We call  $A(x, t)$  a  $\hat{\tau}$ -ball of center  $x$  and radius  $t$ .

We claim that  $\mathcal{B}$  is a basis for a topology on  $M^2$ . In order to prove this let  $A_0$  and  $A_1$  be intersecting sets in  $\mathcal{B}$  and  $x \in A_0 \cap A_1$ . We show that there is  $A = A(x, \varepsilon)$  such that  $A \subseteq A_0 \cap A_1$ .

For any  $y \in M^2$  and  $t > 0$ , let  $A^*(y, t) = A(y, t) \setminus \{y\}$ . Note that, for any  $A \in \mathcal{B}$ , the set  $A^*$  is  $e$ -open.

**Case 1:**  $x \in A_0^* \cap A_1^*$ . Since  $A_0^* \cap A_1^*$  is  $e$ -open there is some  $\varepsilon > 0$  such that  $B_2(x, \varepsilon) \subseteq A_0^* \cap A_1^* \subseteq A_0 \cap A_1$ . Hence we may take  $A = A(x, \varepsilon) \subseteq A_0 \cap A_1$ .

**Case 2:**  $x \notin A_0^* \cap A_1^*$ . Without loss of generality suppose that  $A_0 = A(x, \varepsilon_0)$  for some  $\varepsilon_0 > 0$ . If  $A_1 = A(x, \varepsilon_1)$  for some  $\varepsilon_1 > 0$ , let  $\varepsilon = \min\{\varepsilon_0, \varepsilon_1\}$  and  $A = A(x, \varepsilon)$ . Otherwise, by analogy to case 1, let  $\varepsilon_2 > 0$  be such that  $A(x, \varepsilon_2) \subseteq A_1^*$  and let  $A = A(x, \varepsilon)$ , where  $\varepsilon := \min\{\varepsilon_0, \varepsilon_2\}$ .

So we may conclude that  $\mathcal{B}$  is a basis. Let  $\hat{\tau}$  be the corresponding topology.

Every  $e$ -open set in  $M^2$  is also  $\hat{\tau}$ -open, i.e.  $\tau_e \subsetneq \hat{\tau}$ . In particular  $(M^2, \hat{\tau})$  is Hausdorff. It fails however to be regular, since it is easy to check that, for any  $x \in M^2$  and  $\varepsilon > 0$ ,  $cl_{\hat{\tau}}A(x, \varepsilon) = cl_eB_2(x, \varepsilon)$ , and so, for every  $\hat{\tau}$ -neighborhood  $A$  of  $x$ ,  $cl_{\hat{\tau}}(A) \cap \Gamma_x \neq \emptyset$ . This space, however, is definably separable, which follows from (1) in the following proposition. One may also show that it is definably connected.

**Proposition A.15.** The following are properties of  $(M^2, \hat{\tau})$ .

- (1) Any one-dimensional subspace of  $(M^2, \hat{\tau})$  is affine.
- (2) No two-dimensional subspace of  $(M^2, \hat{\tau})$  is cell-wise euclidean, in particular no two-dimensional subspace of  $(M^2, \hat{\tau})$  is affine.

*Proof.* Statement (2) is obvious from the definition. We prove (1). By Theorem 6.7.1 it suffices to show that  $(M^2, \hat{\tau})$  contains no subspace definably homeomorphic to an interval with the discrete or right half-open interval topology.

Towards a contradiction let  $I \subseteq M$  be an interval and let  $f : (I, \mu) \hookrightarrow (M^2, \hat{\tau})$  be a definable embedding, where  $\mu \in \{\tau_r, \tau_s\}$ . By o-minimality, after restricting  $f$  if necessary, we may assume that  $f$  is an  $e$ -embedding too.

Since  $\mu = \tau_r$  or  $\mu = \tau_s$  and  $f$  is an embedding, it follows that, for any  $t \in I$ ,  $\hat{\tau} - \lim_{s \rightarrow t^-} f(s) \neq f(t)$ . So, by o-minimality, for every  $t \in I$  there exists some  $\varepsilon_t > 0$  and  $s' < t$  such that, for all  $s' < s < t$ ,  $f(s) \notin A(f(t), \varepsilon_t)$ . However, since  $f$  is an  $e$ -embedding, there is also some  $s' < s'' < t$  such that, for all  $s'' < s < t$ ,  $f(s) \in B_2(f(t), \varepsilon_t)$ , and so  $f[(s'', t)] \subseteq \Gamma_{f(t)}$ . For any  $t \in I$ , let  $s_t = \inf\{s \in I : s < t, f[(s, t)] \subseteq \Gamma_{f(t)}\}$ . This family is definable uniformly on  $t \in I$ . By o-minimality, there exists an interval  $J \subseteq I$  such that, for every  $t \in J$ ,  $s_t < J$ . In other words, for every  $s < t$  in  $J$ , it holds that  $f(s) \in \Gamma_{f(t)}$ .

We now claim that, for any two distinct points  $y, z \in M^2$ ,  $|\Gamma_y \cap \Gamma_z| \leq 1$ . In that case, we have a contradiction, since we have shown that, for any  $s, s', t, t' \in J$ , if  $s < s' < t < t'$  then  $\{f(s), f(s')\} \subseteq \Gamma_{f(t)} \cap \Gamma_{f(t')}$ . It therefore remains to prove the claim.

Let  $y = \langle y_1, y_2 \rangle \in M^2$  and  $z = \langle z_1, z_2 \rangle \in M^2$ . Suppose that there exist  $t, t' > 0$  such that

$$\langle y_1 + t, y_2 + t^2 \rangle = \langle z_1 + t', z_2 + t'^2 \rangle.$$

If  $y_1 = z_1$ , then  $t = t'$  and  $y = z$ . If  $y_1 \neq z_1$ , then we substitute  $t' = t + y_1 - z_1$  in  $t^2 = z_2 - y_2 + t'^2$  to get

$$t^2 = z_2 - y_2 + t^2 + 2t(y_1 - z_1) + (y_1 - z_1)^2,$$

and hence

$$t = \frac{y_2 - z_2 - (y_1 - z_1)^2}{2(y_1 - z_1)}.$$

which proves the claim.  $\square$

A natural question to ask is whether we may generalize Theorem 6.7.1 to spaces of all dimensions by substituting the condition of having a definable copy of an interval with the  $\tau_r$  or  $\tau_s$  topologies with simply not being cell-wise euclidean. The answer, even if adding the additional assumption that the space be regular, is no, as witnessed by the following, and final, example.

**Example A.16** (Space that is  $T_3$  and cell-wise euclidean but not definably metrizable). Let  $X = \{\langle x, y \rangle \in M^2 : y \geq 0\}$  be the closed upper half-plane. Let  $\mathcal{B}_e$  be a definable basis for the euclidean topology in  $\{\langle x, y \rangle \in M^2 : y > 0\}$ . For any  $x \in M$  and  $t > 0$ , let

$$A(x, t) = \{\langle x, 0 \rangle\} \cup \{\langle x', y \rangle \in M^2 : |x' - x| < t, 0 \leq y < t|x' - x|\}.$$

Note that, for every  $x \in M$  and  $t > 0$ , there is  $t' > 0$  such that  $A(x, t') \subseteq B_2(x, t)$ , while the converse is not true. Moreover, for every  $x \in M$ , the family  $\{A(x, t) : t > 0\}$  is nested and, for every  $t > 0$ , the set  $A(x, t) \setminus \{\langle x, 0 \rangle\}$  is  $e$ -open in  $X$ . From these three facts it follows, in a manner similar to the case analysis in Example A.14, that the definable family  $\mathcal{B}_{\tilde{\tau}} = \mathcal{B}_e \cup \{A(x, t) : x \in M, t > 0\}$  is a basis for a topology  $\tilde{\tau}$  on  $X$ .

Since we have  $\tau_e|_X \subsetneq \tilde{\tau}$ , the topology  $\tilde{\tau}$  is Hausdorff. Note that, for every  $x \in M$  and  $t > 0$ ,  $cl_{\tilde{\tau}}A(x, t) = cl_eA(x, t)$ , and so  $(X, \tilde{\tau})$  is also regular. Moreover the disjoint subspaces  $\{\langle x, y \rangle : x \in M, y > 0\}$  and  $\{\langle x, y \rangle : x \in M, y = 0\}$  are both euclidean, i.e. the space is cell-wise euclidean. In particular the space is definably separable. Finally, it is also definably connected.

When  $\mathcal{M}$  expands the field of reals this space is separable but not second countable and thus not metrizable. From the completeness of the theory of real closed fields it follows that there is no metric on  $X$  definable in the language of ordered rings that induces  $\tilde{\tau}$ . We show that this holds in greater generality.

**Proposition A.17.** The space  $(X, \tilde{\tau})$  is not definably metrizable.

*Proof.* Towards a contradiction suppose that  $(X, \tilde{\tau})$  is definably metrizable with definable metric  $d$ . For every  $x \in M$  let  $r_x = \sup\{0 < t < 1 : B_d(\langle x, 0 \rangle, t) \cap (\{x\} \times (0, \infty)) = \emptyset\}$ . Note that, by definition of the neighborhoods  $A(x, t)$ , we have that  $r_x > 0$  for every  $x \in M$ . By o-minimality, there exists an interval  $I \subseteq M$  and some  $r > 0$  such that, for every  $x \in I$ ,  $r \leq r_x$ . Now fix  $x \in I$  and consider the  $d$ -ball  $B_d(\langle x, 0 \rangle, r/2)$ . By definition of  $\tilde{\tau}$ , there exists some  $y \in I \setminus \{x\}$  and some  $s > 0$  such that  $\{y\} \times [0, s] \subseteq B_d(\langle x, 0 \rangle, r/2)$ . But then, by the triangle inequality,  $d(\langle y, 0 \rangle, \langle y, s \rangle) \leq d(\langle y, 0 \rangle, \langle x, 0 \rangle) + d(\langle x, 0 \rangle, \langle y, s \rangle) < r \leq r_y$ . This is a contradiction since, for every  $0 < t < r_y$ ,  $B_d(\langle y, 0 \rangle, t) \cap (\{y\} \times (0, \infty)) = \emptyset$ .  $\square$

## VITA

**2016–2017** Wissenschaftliche Hilfskraft (Research Assistant) at Universität Konstanz.

**2014–2015** MAST degree in Mathematics at University of Cambridge.

**2010–2014** BSc degree in Mathematics at Universitat de València.

## REFERENCES

- [1] P. Andújar Guerrero, *Types, transversals and definable compactness in o-minimal structures*, In preparation.
- [2] P. Andújar Guerrero, M. E. M. Thomas, and E. Walsberg, “Directed sets and topological spaces definable in o-minimal structures,” *J. London Math. Soc.*, to appear.
- [3] R. Arefiev, “On the property of monotonicity for weakly o-minimal structures,” in *Algebra and model theory. 2. Collection of papers*, A. Pinus and K. Ponomarev, Eds., Novosibirsk, 1997, pp. 8–15.
- [4] M. Aschenbrenner, A. Dolich, D. Haskell, D. Macpherson, and S. Starchenko, “Vapnik-Chervonenkis density in some theories without the independence property, I,” *Trans. Amer. Math. Soc.*, vol. 368, no. 8, pp. 5889–5949, 2016, ISSN: 0002-9947. DOI: [10.1090/tran/6659](https://doi.org/10.1090/tran/6659).
- [5] M. Aschenbrenner and A. Fischer, “Definable versions of theorems by Kirszbraun and Helly,” *Proc. Lond. Math. Soc. (3)*, vol. 102, no. 3, pp. 468–502, 2011, ISSN: 0024-6115. DOI: [10.1112/plms/pdq029](https://doi.org/10.1112/plms/pdq029).
- [6] M. Aschenbrenner and A. Thamrongthanyalak, “Whitney’s extension problem in o-minimal structures,” *Rev. Mat. Iberoam.*, vol. 35, no. 4, pp. 1027–1052, 2019, ISSN: 0213-2230. DOI: [10.4171/rmi/1077](https://doi.org/10.4171/rmi/1077).
- [7] C. Brech and P. Koszmider, “On universal Banach spaces of density continuum,” *Israel J. Math.*, vol. 190, pp. 93–110, 2012, ISSN: 0021-2172. DOI: [10.1007/s11856-011-0183-5](https://doi.org/10.1007/s11856-011-0183-5).
- [8] A. Chernikov and I. Kaplan, “Forking and dividing in  $NTP_2$  theories,” *J. Symbolic Logic*, vol. 77, no. 1, pp. 1–20, 2012, ISSN: 0022-4812. DOI: [10.2178/jsl/1327068688](https://doi.org/10.2178/jsl/1327068688).
- [9] S. Cotter and S. Starchenko, “Forking in VC-minimal theories,” *J. Symbolic Logic*, vol. 77, no. 4, pp. 1257–1271, 2012, ISSN: 0022-4812. DOI: [10.2178/jsl.7704110](https://doi.org/10.2178/jsl.7704110).
- [10] J. Dieudonné, *Foundations of modern analysis*. Academic Press, New York-London, 1969, pp. xviii+387, Enlarged and corrected printing, Pure and Applied Mathematics, Vol. 10-I.
- [11] P. Dodos and V. Ferenczi, “Some strongly bounded classes of Banach spaces,” *Fund. Math.*, vol. 193, no. 2, pp. 171–179, 2007, ISSN: 0016-2736. DOI: [10.4064/fm193-2-5](https://doi.org/10.4064/fm193-2-5).
- [12] A. Dolich, “Forking and independence in o-minimal theories,” *J. Symbolic Logic*, vol. 69, no. 1, pp. 215–240, 2004, ISSN: 0022-4812. DOI: [10.2178/jsl/1080938838](https://doi.org/10.2178/jsl/1080938838).

- [13] A. Dolich, C. Miller, and C. Steinhorn, “Structures having o-minimal open core,” *Transactions of The American Mathematical Society*, vol. 362, no. 3, pp. 1371–1411, 2010. DOI: [10.1090/S0002-9947-09-04908-3](https://doi.org/10.1090/S0002-9947-09-04908-3).
- [14] L. van den Dries, “A generalization of the Tarski-Seidenberg theorem, and some non-definability results,” *Bull. Amer. Math. Soc. (N.S.)*, vol. 15, no. 2, pp. 189–193, 1986, ISSN: 0273-0979. DOI: [10.1090/S0273-0979-1986-15468-6](https://doi.org/10.1090/S0273-0979-1986-15468-6).
- [15] L. van den Dries, “Limit sets in o-minimal structures,” in *O-minimal Structures, Lisbon 2003*, ser. Proceedings of a Summer School by the European Research and Training Network, RAAG, 2005, pp. 172–215.
- [16] L. van den Dries, “Remarks on Tarski’s problem concerning  $(\mathbf{R}, +, \cdot, \exp)$ ,” in *Logic Colloquium ’82 (Florence, 1982)*, ser. Stud. Logic Found. Math. Vol. 112, North-Holland, Amsterdam, 1984, pp. 97–121. DOI: [10.1016/S0049-237X\(08\)71811-1](https://doi.org/10.1016/S0049-237X(08)71811-1).
- [17] L. van den Dries, *Tame Topology and O-minimal Structures*, ser. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1998, vol. 248. DOI: [10.1017/CBO9780511525919](https://doi.org/10.1017/CBO9780511525919).
- [18] L. van den Dries and A. H. Lewenberg, “ $T$ -convexity and tame extensions,” *J. Symbolic Logic*, vol. 60, no. 1, pp. 74–102, 1995.
- [19] L. van den Dries and C. Miller, “On the real exponential field with restricted analytic functions,” *Israel J. Math.*, vol. 85, no. 1-3, pp. 19–56, 1994, ISSN: 0021-2172. DOI: [10.1007/BF02758635](https://doi.org/10.1007/BF02758635).
- [20] M. J. Edmundo, “Structure theorems for o-minimal expansions of groups,” *Annals of Pure and Applied Logic*, vol. 102, no. 1-2, pp. 159–181, 2000. DOI: [10.1016/S0168-0072\(99\)00043-3](https://doi.org/10.1016/S0168-0072(99)00043-3).
- [21] A. Farhat, “The basis problem for subspaces of monotonically normal compacta,” *Topology Appl.*, vol. 195, pp. 107–118, 2015, ISSN: 0166-8641. DOI: [10.1016/j.topol.2015.09.028](https://doi.org/10.1016/j.topol.2015.09.028). [Online]. Available: <https://doi.org/10.1016/j.topol.2015.09.028>.
- [22] J. Flum and M. Ziegler, *Topological model theory*, ser. 769. Springer-Verlag, Berlin Heidelberg, 1980. DOI: [10.1007/BFb0097006](https://doi.org/10.1007/BFb0097006).
- [23] A. Fornasiero, “Definable compactness for topological structures,” In preparation.
- [24] A. Grothendieck, “Esquisse d’un programme,” in *Geometric Galois actions, 1*, ser. London Math. Soc. Lecture Note Ser. Vol. 242, With an English translation on pp. 243–283, Cambridge Univ. Press, Cambridge, 1997, pp. 5–48.

- [25] G. Gruenhage, “On the existence of metrizable or Sorgenfrey subspaces,” in *General topology and its relations to modern analysis and algebra, VI (Prague, 1986)*, ser. Res. Exp. Math. Vol. 16, Heldermann, Berlin, 1988, pp. 223–230.
- [26] H. R. Johnson and M. C. Laskowski, “Compression schemes, stable definable families, and o-minimal structures,” *Discrete Comput. Geom.*, vol. 43, no. 4, pp. 914–926, 2010, ISSN: 0179-5376. DOI: [10.1007/s00454-009-9201-3](https://doi.org/10.1007/s00454-009-9201-3).
- [27] W. Johnson, “Interpretable sets in dense o-minimal structures,” *J. Symbolic Logic*, vol. 83, no. 4, pp. 1477–1500, 2018, ISSN: 0022-4812. DOI: [10.1017/jsl.2018.50](https://doi.org/10.1017/jsl.2018.50).
- [28] M. Karpinski and A. Macintyre, “Approximating volumes and integrals in o-minimal and  $p$ -minimal theories,” in *Connections between model theory and algebraic and analytic geometry*, ser. Quad. Mat. Vol. 6, Dept. Math., Seconda Univ. Napoli, Caserta, 2000, pp. 149–177.
- [29] J. F. Knight, A. Pillay, and C. Steinhorn, “Definable sets in ordered structures. II,” *Trans. Amer. Math. Soc.*, vol. 295, no. 2, pp. 593–605, 1986, ISSN: 0002-9947. DOI: [10.2307/2000053](https://doi.org/10.2307/2000053).
- [30] M. C. Laskowski and C. Steinhorn, “On o-minimal expansions of Archimedean ordered groups,” *J. Symbolic Logic*, vol. 60, no. 3, pp. 817–831, 1995. DOI: [10.2307/2275758](https://doi.org/10.2307/2275758).
- [31] D. Marker and C. I. Steinhorn, “Definable types in  $O$ -minimal theories,” *J. Symbolic Logic*, vol. 59, no. 1, pp. 185–198, 1994, ISSN: 0022-4812. DOI: [10.2307/2275260](https://doi.org/10.2307/2275260).
- [32] J. Matoušek, “Bounded VC-dimension implies a fractional Helly theorem,” *Discrete Comput. Geom.*, vol. 31, no. 2, pp. 251–255, 2004, ISSN: 0179-5376. DOI: [10.1007/s00454-003-2859-z](https://doi.org/10.1007/s00454-003-2859-z). [Online]. Available: <https://doi.org/10.1007/s00454-003-2859-z>.
- [33] L. Mirsky, “A dual of Dilworth’s decomposition theorem,” *Amer. Math. Monthly*, vol. 78, pp. 876–877, 1971, ISSN: 0002-9890. DOI: [10.2307/2316481](https://doi.org/10.2307/2316481).
- [34] J. T. Moore, “A solution to the  $L$  space problem,” *J. Amer. Math. Soc.*, vol. 19, no. 3, pp. 717–736, 2006, ISSN: 0894-0347. DOI: [10.1090/S0894-0347-05-00517-5](https://doi.org/10.1090/S0894-0347-05-00517-5). [Online]. Available: <https://doi-org.ezproxy.lib.purdue.edu/10.1090/S0894-0347-05-00517-5>.
- [35] Y. Peterzil and A. Pillay, “Generic sets in definably compact groups,” *Fund. Math.*, vol. 193, no. 2, pp. 153–170, 2007, ISSN: 0016-2736. DOI: [10.4064/fm193-2-4](https://doi.org/10.4064/fm193-2-4).
- [36] Y. Peterzil and A. Rosel, “Definable one-dimensional topologies in o-minimal structures,” *Arch. Math. Logic*, vol. 59, no. 1-2, pp. 103–125, 2020, ISSN: 0933-5846. DOI: [10.1007/s00153-019-00680-z](https://doi.org/10.1007/s00153-019-00680-z).



- [37] Y. Peterzil and C. Steinhorn, “Definable compactness and definable subgroups of  $o$ -minimal groups,” *J. London Math. Soc. (2)*, vol. 59, no. 3, pp. 769–786, 1999.
- [38] A. Pillay, “First order topological structures and theories,” *J. Symbolic Logic*, vol. 52, no. 3, pp. 763–778, 1987.
- [39] A. Pillay, “On groups and fields definable in  $o$ -minimal structures,” *J. Pure Appl. Algebra*, vol. 53, no. 3, pp. 239–255, 1988, ISSN: 0022-4049. DOI: [10.1016/0022-4049\(88\)90125-9](https://doi.org/10.1016/0022-4049(88)90125-9). [Online]. Available: [https://doi.org/10.1016/0022-4049\(88\)90125-9](https://doi.org/10.1016/0022-4049(88)90125-9).
- [40] A. Pillay and C. Steinhorn, “Definable sets in ordered structures. I,” *Trans. Amer. Math. Soc.*, vol. 295, no. 2, pp. 565–592, 1986, ISSN: 0002-9947. DOI: [10.2307/2000052](https://doi.org/10.2307/2000052).
- [41] A. Pillay and C. Steinhorn, “On Dedekind complete  $o$ -minimal structures,” *J. Symbolic Logic*, vol. 52, no. 1, pp. 156–164, 1987, ISSN: 0022-4812. DOI: [10.2307/2273870](https://doi.org/10.2307/2273870).
- [42] J. Ramakrishnan, “Definable linear orders definably embed into lexicographic orders in  $o$ -minimal structures,” *Proc. Amer. Math. Soc.*, vol. 141, no. 5, arXiv:1003.5400, pp. 1809–1819, 2013. DOI: [10.1090/S0002-9939-2012-11424-8](https://doi.org/10.1090/S0002-9939-2012-11424-8). [Online]. Available: <https://doi.org/10.1090/S0002-9939-2012-11424-8>.
- [43] J. Ramakrishnan and C. Steinhorn, “Definably extending partial orders in totally ordered structures,” *MLQ Math. Log. Q.*, vol. 60, no. 3, pp. 205–210, 2014. DOI: [10.1002/malq.201300047](https://doi.org/10.1002/malq.201300047).
- [44] R. Robson, “Embedding semi-algebraic spaces,” *Math. Z.*, vol. 183, no. 3, pp. 365–370, 1983, ISSN: 0025-5874. DOI: [10.1007/BF01176477](https://doi.org/10.1007/BF01176477).
- [45] N. Sauer, “On the density of families of sets,” *J. Combinatorial Theory Ser. A*, vol. 13, pp. 145–147, 1972, ISSN: 0097-3165. DOI: [10.1016/0097-3165\(72\)90019-2](https://doi.org/10.1016/0097-3165(72)90019-2).
- [46] P. Simon, “Invariant types in NIP theories,” *J. Math. Log.*, vol. 15, no. 2, pp. 1550006, 26, 2015, ISSN: 0219-0613. DOI: [10.1142/S0219061315500063](https://doi.org/10.1142/S0219061315500063).
- [47] P. Simon and S. Starchenko, “On forking and definability of types in some DP-minimal theories,” *J. Symbolic Logic*, vol. 79, no. 4, pp. 1020–1024, 2014, ISSN: 0022-4812. DOI: [10.1017/jsl.2014.45](https://doi.org/10.1017/jsl.2014.45).
- [48] M. E. M. Thomas, “Convergence results for function spaces over  $o$ -minimal structures,” *J. Log. Anal.*, vol. 4, no. 1, 2012. DOI: [10.4115/jla.2012.4.1](https://doi.org/10.4115/jla.2012.4.1).
- [49] E. Walsberg, “On the topology of metric spaces definable in  $o$ -minimal expansions of fields,” *arXiv e-prints*, 2015, arXiv:1510.07291.

- [50] E. Walsberg, “The Marker-Steinhorn theorem via definable linear orders,” *Notre Dame J. Form. Log.*, vol. 60, no. 4, pp. 701–706, 2019, ISSN: 0029-4527. DOI: [10.1215/00294527-2019-0026](https://doi.org/10.1215/00294527-2019-0026).
- [51] A. J. Wilkie, “Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function,” *J. Amer. Math. Soc.*, vol. 9, no. 4, pp. 1051–1094, 1996, ISSN: 0894-0347. DOI: [10.1090/S0894-0347-96-00216-0](https://doi.org/10.1090/S0894-0347-96-00216-0).